Topics: real numbers, intervals, inequalities

5.1 (Bounds on intervals)**.** Let a and b be real numbers with $a < b$. For each interval below, determine (if they exist) the maximum, minimum, infimum, and supremum in $\mathbf{R}$.

(Recall the definitions of maximum and minimum from Tutorial Question 4.9.)

Interval	Infimum	Supremum	Minimum	Maximum
(a, b)				
$[a, b]$				
$(a, b]$				
$[a, b)$				
(a, ∞)				
$[a, \infty)$				
$(-\infty, b)$				
$(-\infty, b]$				

Solution. Here is the completed table:

Interval	Infimum	Supremum	Minimum	Maximum
(a, b)	a	b	none	none
$[a, b]$	a	b	a	b
$(a, b]$	a	b	none	b
$[a, b)$	a	b	a	none
(a, ∞)	a	none	none	none
$[a, \infty)$	a	none	a	none
$(-\infty, b)$	none	b	none	none
$(-\infty, b]$	none	b	none	b

$\square$

5.2 (An interesting real number)**.** For any $n \in \mathbf{N}$, let $f(n)$ be the following sum:

$$f(n) = \sum_{k=0}^{n} \frac{1}{k!}.$$

(Recall that $0! = 1$.)

(a) Compute $f(0), f(1), f(2), f(10), f(11), f(48), f(49)$, and $f(50)$.

[**Hint**: Use Wolfram Alpha to do this quickly: `sum 1/k!  with k from 0 to n`.]

(b) Consider the following set

$$A = \{x \in \mathbf{Q} : \ (\exists n \in \mathbf{N}) \ x < f(n)\}.$$

Show that if $x \in A$, then there exists a rational number $y \in A$ such that $x < y$.

(c) Can you guess which number $\sup A$ is? Do you think it is rational or irrational?

(**Note:** We do not yet know how to prove that this set is even bounded above, let alone how to compute its supremum! You may assume the set is bounded above, and make a guess of the supremum based purely on your calculations in Part (a).)

Solution.

(a)

$$\begin{aligned} f(0) &= \frac{1}{0!} = \frac{1}{1} = 1, \\ f(1) &= 1 + \frac{1}{1!} = 1 + 1 = 2, \\ f(2) &= 2 + \frac{1}{2!} = 2 + \frac{1}{2} = \frac{5}{2}. \end{aligned}$$

For larger n, such as $f(10), f(11), f(48), f(49), f(50)$, these are rational but have large numerators and denominators. For example,

$$f(10) = \sum_{k=0}^{10} \frac{1}{k!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040}+\frac{1}{40320}+\frac{1}{362880}+\frac{1}{3628800} = \frac{9864101}{3628800}.$$

The decimal approximations are:

$$\begin{aligned} f(10) &\approx 2.718281801, \\ f(11) &\approx 2.718281826, \\ f(48) &\approx 2.718281828459045, \\ f(49) &\approx 2.718281828459045, \\ f(50) &\approx 2.718281828459045. \end{aligned}$$

(b) By definition, there exists n such that $x < f(n)$. Let $y = f(n)$. Then y is rational (as a finite sum of rational numbers), $x < y$, and $y < f(n+1)$ (because $f(n+1) = f(n) + 1/(n+1)!$). Thus there exists a natural number m (namely $m = n+1$) such that $y < f(m)$. This means that $y \in A$.

(c) Notice that starting around $f(48)$, the sum is close to $e \approx 2.718281828459045...$. Adding more terms after $n = 50$ changes the decimal only after the 16th digit. A guess supported by the numbers in (a) is that the supremum is e, the base of the natural logarithm. ☐

5.3 (Geometry of inequalities)**.** For each inequality, find all values of x that make the inequality True, and illustrate these values on the real line.

(a) $|x| < 4$

(b) $|x+1| < 4$

(c) $|x-1| < 4$

(d) $|2x| < 4$

(e) $|x/2| < 4$

(f) $|2x + 10| < 4$

(g) $|-2x + 10| < 4$.

Solution. Recall for any real number a and $b > 0$, the inequality $|a| < b$ is equivalent to

$$-b < a < b.$$

(a) $|x| < 4$

$$-4 < x < 4 \quad \Rightarrow \quad x \in (-4, 4).$$

(b) $|x + 1| < 4$

$$-4 < x + 1 < 4 \quad \Rightarrow \quad -5 < x < 3 \quad \Rightarrow \quad x \in (-5, 3).$$

(c) $|x - 1| < 4$

$$-4 < x - 1 < 4 \quad \Rightarrow \quad -3 < x < 5 \quad \Rightarrow \quad x \in (-3, 5).$$

(d) $|2x| < 4$

$$-4 < 2x < 4 \quad \Rightarrow \quad -2 < x < 2 \quad \Rightarrow \quad x \in (-2, 2).$$

(e) $\left|\frac{x}{2}\right| < 4$

$$-4 < \frac{x}{2} < 4 \quad \Rightarrow \quad -8 < x < 8 \quad \Rightarrow \quad x \in (-8, 8).$$

(f) $|2x + 10| < 4$

$$-4 < 2x + 10 < 4 \quad \Rightarrow \quad -14 < 2x < -6 \quad \Rightarrow \quad -7 < x < -3 \quad \Rightarrow \quad x \in (-7, -3).$$

(g) $|-2x + 10| < 4$

$$-4 < -2x + 10 < 4 \quad \Rightarrow \quad 6 < -2x < 14 \quad \Rightarrow \quad -7 < x < -3 \quad \Rightarrow \quad x \in (-7, -3).$$

☐

5.4 (Inequalities and transformations)**.** At some point in the past you have learned about geometric transformations (translation, reflection, scaling, etc.) of the plane, and their effects on the graphs of functions.

For example, given a function f we can draw conclusions about the shape of $f(2x + 1) - 3$ using words like **scale** or **translate**.

When you compare your drawings of the solutions in the previous question, perhaps we can see the same sort of behaviour, this time related to geometric transformations of the line.

Let $a, b, c \in \mathbf{R}$, with $a, c > 0$. In answering the following questions, use geometric language such as

(a) What transformation turns the set of solutions for $|x| < c$ into the set of solutions for $|x + b| < c$?

(b) What transformation turns the set of solutions for $|x| < c$ into the set of solutions for $|ax| < c$?

(c) What transformation turns the set of solutions for $|ax + b| < c$ into the set of solutions $|-ax + b| < c$?

(d) What transformation turns the set of solutions for $|ax + b| < c$ into the set of solutions for $|ax + b| \geqslant c$? Can you express this transformation using the language of set theory?

Solution.

(a) translation;

(b) scaling;

(c) reflection;

(d) complement. ☐

5.5 (Probing the triangle inequality)**.** Give

(a) an example of $x, y \in \mathbf{R}$ such that the triangle inequality is in fact an equality;

(b) an example of $x, y \in \mathbf{R}$ such that the triangle inequality is a strict inequality.

Guided by your examples, complete the following into a `True` statement: "given $x, y \in \mathbf{R}$, the triangle inequality for $|x + y|$ is an equality if and only if . . . "

Solution.

(a) For $x = 1, y = 2$ we have:

$$|x + y| = |3| = 3 = 1 + 2 = |1| + |2| = |x| + |y|.$$

(b) For $x = 1$ and $y = -1$ we have:

$$|x + y| = |0| = 0 < 1 + 1 = |1| + |-1| = |x| + |y|.$$

☐

5.6 (Neighbourhoods)**.** Let $c, \varepsilon \in \mathbf{R}$ with $\varepsilon > 0$. The *ε-neighbourhood of c* is by definition the interval $(c - \varepsilon, c + \varepsilon)$.

(a) Draw, on the real line, the 0.1-neighbourhood of 2.

(b) Let $x \in \mathbf{R}$. Prove that x is in the ε-neighbourhood of c if and only if $|x - c| < \varepsilon$.

(c) Prove that the ε-neighbourhood of 2 contains a real number other than 2.

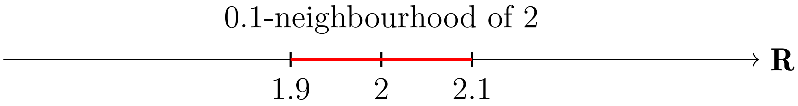
[**Hint**: Use the Archimedean Principle.]

Solution.

(a) The 0.1-neighbourhood of 2 is the interval

$$(2 - 0.1, 2 + 0.1) = (1.9, 2.1).$$

A simple real-line illustration:



(b) Let $x \in \mathbf{R}$. By definition, x is in the ε-neighbourhood of c if and only if

$$c - \varepsilon < x < c + \varepsilon.$$

Subtracting c from all sides gives $-\varepsilon < x - c < \varepsilon$, which is equivalent to $|x - c| < \varepsilon$.

Conversely, if $|x - c| < \varepsilon$, then $-\varepsilon < x - c < \varepsilon$, which implies $c - \varepsilon < x < c + \varepsilon$. Hence, x is in the ε-neighbourhood of c if and only if $|x - c| < \varepsilon$.

(c) We are looking for $x \in \mathbf{R}$ such that $|x - 2| < \varepsilon$.

By the Archimedean Principle III, for any $x, y \in \mathbf{R}$ with $x, y > 0$ there exists $n \in \mathrm{N}$ so that $y < nx$. Apply the Archimedean Principle III to ε and 1 to obtain $n \in \mathbf{N}$ such that $1 < n\varepsilon$, so $\frac{1}{n} < \varepsilon$.

Let $x = 2 + 1/n$. By construction we have

$$|x - 2| = |2 + 1/n - 2| = 1/n < \varepsilon.$$

Therefore $x = 2 + 1/n \neq 2$ is in the ε-neighbourhood of 2. $\square$

Topics: bounds, limits of sequences

5.7 (Bounding functions)**.** To *bound* a function $f:\ [a,b] \longrightarrow \mathbf{R}$ on the interval $[a,b]$ means to find a real number K such that $|f(x)| \leqslant K$ for all $x \in [a,b]$.

(a) Bound the function $f:\ [-2,4] \longrightarrow \mathbf{R}$ given by $f(x) = |x-3|$, on $[-2,4]$.

(b) Bound

$$\frac{1}{|x-3|}$$

for $x \in (-2,0)$.

Can it be bounded on $[-2,4] \setminus \{3\} = [-2,3) \cup (3,4]$?

(c) Bound the function $f:\ (-2,7) \longrightarrow \mathbf{R}$ given by

$$f(x) = 2x^2 - 6x + 7.$$

[**Hint**: Any upper bound on $|f|$ will do! The triangle inequality may be useful here.]

Solution.

(a) As $-2 \leqslant x \leqslant 4$, we have

$$-5 \leqslant x-3 \leqslant 1 \quad \Rightarrow \quad |x-3| \leqslant |-5| = 5.$$

(b) For $-2 < x < 0$,

$$-5 < x-3 < -3 \quad \Rightarrow \quad -\frac{1}{3} < \frac{1}{x-3} < -\frac{1}{5} \quad \Rightarrow \quad \frac{1}{|x-3|} < \left|-\frac{1}{3}\right| = \frac{1}{3}.$$

Thus, $\frac{1}{|x-3|}$ is bounded by $\frac{1}{3}$ on the interval $(-2,0)$.

If $x \in [-2,4] \setminus \{3\}$, then, $-2 \leqslant x \leqslant 4$, and $x \neq 3$, we have $-5 \leqslant x-3 \leqslant 1$, and $0 < |x-3| \leqslant 5$.

Since $\frac{1}{|x-3|}$ blows up near $x = 3$, the function is **not** bounded on the interval $[-2,4]\setminus\{3\}$.

(c) We look for $K \geqslant |f(x)|$ on $-2 < x < 7$. By the triangle inequality (applied twice),

$$\left|2x^2 - 6x + 7\right| \leqslant |2x^2| + |6x| + |7| = 2|x|^2 + 6|x| + 7 \leqslant 2 \times 7^2 + 6 \times 7 + 7 = 147. \quad \square$$

5.8 (Proving a limit (or five))**.**

(a) Look up Definition 3.6 and copy it down or have it handy throughout this question.

(b) Make a guess L about the limit of the sequence

$$(x_n) = \left(\frac{n^2}{3n^2-4}\right).$$

(c) Letting L denote your guess from the previous part, let $p(n,\varepsilon)$ be the condition

$$|x_n - L| < \varepsilon.$$

Find a value $M \in \mathbf{N}$ such that $p(n, 0.01)$ is True for all $n > M$.

(d) Generalising your work in the previous part, find a formula for M as a function of ε so that $p(n, \varepsilon)$ is `True` for all $n > M$.

(e) Write an informal proof that

$$\lim_{n\to\infty} \frac{n^2}{3n^2-4} = L,$$

where L is your guessed limit. (It's $1/3$, isn't it?)

Your proof should have the following structure:

> Let $\varepsilon > 0$. Let $M =$ (your formula from the previous part).
> Suppose $n > M$.
>
> (Algebraic manipulations as necessary.)
>
> We conclude that $|x_n - L| < \varepsilon$ for all $n > M$.
> As this holds for all $\varepsilon > 0$, it follows that
>
> $$\lim_{n\to\infty} \frac{n^2}{3n^2-4} = L.$$

(f) Was that fun? Then repeat the process in parts (b) to (e) with the sequences

$$\begin{aligned}(y_n) &= \left(\frac{1}{\sqrt{n}} + 1\right)\\ (z_n) &= \left(\frac{6}{3n^2-4}\right)\\ (w_n) &= \left(\frac{(-1)^n}{2n}\right)\\ (u_n) &= \big(1, 1, 1, 1, \dots\big).\end{aligned}$$

Solution.

(a) Let (x_n) be a sequence and let $L \in \mathbf{R}$.

We say that (x_n) *converges to L* if: for every $\varepsilon > 0$ there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, if $n > M$ then $|x_n - L| < \varepsilon$.

In this case, we write

$$x_n \longrightarrow L \qquad \text{or} \qquad \lim_{n\to\infty} x_n = L$$

and call L a *limit* of the sequence (x_n).

(b) $L = 1/3$.

(c)

$$\begin{aligned}\left|\frac{n^2}{3n^2-4} - \frac{1}{3}\right| &< 0.01\\ \left|n^2 - \frac{3n^2-4}{3}\right| &< 0.01\,(3n^2-4)\\ \frac{4}{3} &< 0.01\,(3n^2-4)\\ \sqrt{\frac{4}{9(0.01)} + \frac{4}{3}} &< n.\end{aligned}$$

Now we take M to be any integer larger than the number on the left hand side of the last inequality, for instance $M = 7$.

(d)

$$\begin{aligned} \left|\frac{n^2}{3n^2-4} - \frac{1}{3}\right| &< \varepsilon \\ \left|n^2 - \frac{3n^2-4}{3}\right| &< \varepsilon(3n^2-4) \\ \frac{4}{3} &< \varepsilon(3n^2-4) \\ \sqrt{\frac{4}{9\varepsilon}+\frac{4}{3}} &< n. \end{aligned}$$

So we take M to be any integer larger than (for instance, the ceiling of)

$$\sqrt{\frac{4}{9\varepsilon}+\frac{4}{3}}.$$

(e) Let $\varepsilon > 0$. Let

$$M = \left\lceil \sqrt{\frac{4}{9\varepsilon}+\frac{4}{3}} \right\rceil.$$

Suppose $n > M$.

Then

$$n > \sqrt{\frac{4}{9\varepsilon}+\frac{4}{3}}.$$

Manipulating, we find:

$$\begin{aligned} \sqrt{\frac{4}{9\varepsilon}+\frac{4}{3}} &< n \\ \frac{4}{3} &< \varepsilon(3n^2-4) \\ \left|n^2 - \frac{3n^2-4}{3}\right| &< \varepsilon(3n^2-4) \\ \left|\frac{n^2}{3n^2-4} - \frac{1}{3}\right| &< \varepsilon. \end{aligned}$$

We conclude that $|x_n - 1/3| < \varepsilon$ for all $n > M$.

As this holds for all $\varepsilon > 0$, it follows that

$$\lim_{n\to\infty} \frac{n^2}{3n^2-4} = \frac{1}{3}.$$

(f) For the other sequences we have the limits $y_n \longrightarrow 1$, $z_n \longrightarrow 0$, $w_n \longrightarrow 0$, $u_n \longrightarrow 1$. $\square$

5.9. [Limits are unique]

(a) Suppose $a \in \mathbf{R}$ is such that $|a| < \varepsilon$ for all $\varepsilon > 0$. Then $a = 0$.

(b) Any sequence (x_n) has at most one limit.

Solution.

(a) We proceed by contradiction. Suppose $a \neq 0$, then $|a| > 0$. Let $\varepsilon = \frac{|a|}{2}$, then $\varepsilon > 0$ and $\varepsilon < |a|$, contradiction.

(b) Suppose L and L' are two limits of a sequence (x_n).

Let $\varepsilon > 0$. There exist $M, M' \in \mathbf{N}$ such that

$$\begin{aligned} n > M &\Rightarrow |x_n - L| < \frac{\varepsilon}{2} \\ n > M' &\Rightarrow |x_n - L'| < \frac{\varepsilon}{2}. \end{aligned}$$

Let $n = \mathsf{max}\,\{M, M'\} + 1$, then

$$|L - L'| = |(L - x_n) + (x_n - L')| \leqslant |L - x_n| + |x_n - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we conclude by Part (a) that $|L - L'| = 0$ so $L = L'$. ☐

5.10. [Limits and inequalities] In Theorem 3.16 we have seen that if (x_n) and (y_n) are convergent and $x_n \leqslant y_n$ for all $n \in \mathbf{N}$, then

$$\lim_{n\to\infty} x_n \leqslant \lim_{n\to\infty} y_n.$$

Suppose now that (x_n) and (y_n) are convergent sequences and $x_n < y_n$ for all $n \in \mathbf{N}$. Does it follow that

$$\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n?$$

If you think the answer is Yes, give a proof.

If you think the answer is No, give a counterexample.

Solution. The answer is No.

Let (x_n) be the constant sequence $(0, 0, 0, \dots)$ and let $(y_n) = \left(\frac{1}{n+1}\right)$.

Then $x_n < y_n$ for all $n \in \mathbf{N}$, but both limits are 0, hence the claimed inequality would be $0 < 0$, which is False. ☐