

Topics: convergence of sequences

6.1 (Algebra of Limits: examples). You are told that

$$\left(\frac{1}{n}\right) \longrightarrow 0 \quad \text{and} \quad (1, 1, 1, 1, 1, \dots) \longrightarrow 1.$$

Apply the Algebra of Limits Theorem and the two facts given above to find the limit of each of the following sequences:

(a) $(x_n) = \left(\frac{n^2 + 6}{3n^2 - 4}\right);$

(b) $(y_n) = (3, 3, 3, 3, \dots).$

Solution.

(a) Consider $x_n = \frac{n^2+6}{3n^2-4}$. Divide numerator and denominator by n^2 : $x_n = \frac{1+\frac{6}{n^2}}{3-\frac{4}{n^2}}$. Since $\frac{1}{n} \longrightarrow 0$, we have $\frac{6}{n^2} \longrightarrow 0$. Hence $\frac{6}{n^2} \longrightarrow 0$ and $\frac{4}{n^2} \longrightarrow 0$. By the Algebra of Limits,

$$1 + \frac{6}{n^2} \longrightarrow 1, \quad 3 - \frac{4}{n^2} \longrightarrow 3.$$

Since the denominator limit is nonzero, we can apply the quotient rule: $x_n \longrightarrow \frac{1}{3}$.

(b) The sequence $y_n = (3, 3, 3, \dots)$ can be written as

$$y_n = (1, 1, 1, \dots) + (1, 1, 1, \dots) + (1, 1, 1, \dots).$$

Apply the Algebra of Limits Theorem, we have $y_n \longrightarrow 1 + 1 + 1 = 3$. □

6.2 (Comparisons are your Friend). Familiarise yourself with the statement of the Sandwich Theorem, see [Exercise 3.7](#).

(a) Find an inequality relation between n^2 and $n^2 + 6n + 11$ that is valid for all $n \in \mathbf{N}$. Use this to find

$$\lim_{n \rightarrow \infty} \frac{1}{n^2 + 6n + 11}.$$

(b) Modify the approach in part (a) appropriately so you can find

$$\lim_{n \rightarrow \infty} \frac{1}{n^2 - 6n + 4}.$$

(c) Find the limit of the sequence (x_n) given by

$$(x_n)_{n \geq 1} = \left(\frac{\sin(n)}{n}\right)_{n \geq 1}.$$

Solution.

(a) For all $n \in \mathbf{N}$, $n^2 < n^2 + 6n + 11$. So there is

$$0 < \frac{1}{n^2 + 6n + 11} < \frac{1}{n^2}.$$

Since 0 and $\frac{1}{n^2}$ both converge to 0, by the Sandwich Theorem, $\left(\frac{1}{n^2 + 6n + 11}\right)$ also converges to 0.

(b) We have that $n^2 - 6n + 4 = n(n - 6) + 4$. So for all $n > 6$, we have that

$$n^2 - 6n + 4 = n(n - 6) + 4 > n(n - 6) > n.$$

So for $n > 6$,

$$0 < \frac{1}{n^2 - 6n + 4} < \frac{1}{n}.$$

Since 0 and $\frac{1}{n}$ both converge to 0, by the Sandwich Theorem, so does $\frac{1}{n^2 - 6n + 4}$.

(c) Using the fact that

$$-1 \leq \sin(n) \leq 1,$$

we see that

$$0 = \lim_{n \rightarrow \infty} \frac{-1}{n} \leq \lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

So by the Sandwich Theorem, $(x_n) \rightarrow 0$. □

6.3 (A more intricate limit). The final objective of this question is to determine (with proof) the limit of the sequence

$$(x_n)_{n \geq 1} = (n^{1/n})_{n \geq 1}.$$

(a) Let (y_n) be a sequence and suppose that $y_n^2 \rightarrow 0$. Prove that (y_n) converges and $y_n \rightarrow 0$.

Would your argument work if we had instead $y_n^2 \rightarrow 1$?

(b) From now on we consider the sequence (x_n) defined at the beginning of the question. Prove that $1 \leq x_n$ for all $n \geq 1$.

[**Hint:** First use induction to show that if t is a real number with $0 < t < 1$, then $0 < t^n < 1$ for all $n \geq 1$.]

(c) For any $n \geq 1$, let $y_n = x_n - 1$, so that $x_n = 1 + y_n$.

Use the Binomial Formula

$$(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j \quad \text{for all } a, b \in \mathbf{R}, n \in \mathbf{N}, n \geq 1,$$

to show that $2 > (n - 1)y_n^2$ for all $n \geq 2$.

[**Hint:** Apply the Binomial Formula to $(1 + y_n)^n$ and pick out the term corresponding to $j = 2$.]

(d) Use the Sandwich Theorem to find the limit of (y_n^2) as $n \rightarrow \infty$.

(e) Find the limit of (x_n) as $n \rightarrow \infty$.

Solution.

(a) Let $\varepsilon > 0$. Since $y_n^2 \rightarrow 0$, there exists $M \in \mathbf{N}$ such that if $n > M$ we have

$$|y_n^2 - 0| < \varepsilon^2 \quad \Rightarrow \quad |y_n|^2 < \varepsilon^2 \quad \Rightarrow \quad |y_n| < \varepsilon.$$

We conclude that $y_n \rightarrow 0$.

The argument does not work at all if $y_n^2 \rightarrow 1$, because the sequence $(y_n) = ((-1)^n)$ satisfies this condition but diverges.

- (b) Let t be a real number with $0 < t < 1$.

Base case $n = 1$: $0 < t < 1$ by assumption.

Induction step: let $k \geq 1$ be arbitrary but fixed and suppose that $0 < t^k < 1$. Since $t > 0$ we can multiply these inequalities by t and get $0 < t^{k+1} < t < 1$, which is the statement for $k + 1$.

Now we consider $(x_n) = (n^{1/n})_{n \geq 1}$. Fix $n \geq 1$ and let $t = n^{1/n}$. Suppose that $t < 1$, then by the above we have $t^n < 1$, but $t^n = 1$, contradiction. Therefore $n^{1/n} = t \geq 1$ for all $n \geq 1$.

- (c) For $n \geq 2$, we have

$$n = x_n^n = (1 + y_n)^n = 1 + ny_n + \binom{n}{2}y_n^2 + \sum_{j=3}^n \binom{n}{j}y_n^j > \binom{n}{2}y_n^2 = \frac{n(n-1)}{2}y_n^2.$$

Therefore $2 > (n-1)y_n^2$ for all $n \geq 2$.

- (d) For $n \geq 2$ we have

$$0 \leq y_n^2 < \frac{2}{n-1},$$

so by the Sandwich Theorem we conclude that $y_n^2 \rightarrow 0$.

- (e) By part (a) we have $y_n \rightarrow 0$, hence $x_n = 1 + y_n \rightarrow 1 + 0 = 1$. □

6.4 (Divergent Sequences). Recall the precise statement of [Theorem 3.22](#).

For each sequence below, use the Theorem to determine if the sequence diverges. Caution! There may be some sequences below for which the Theorem gives us no information.

- (a) $(a_n) = (\sqrt{n})$;

- (b) (b_n) given by

$$b_n = \begin{cases} 1 & n = 3k \\ \frac{1}{n} & n = 3k + 1 \\ -\frac{1}{2n} & n = 3k + 2; \end{cases}$$

- (c) (c_n) given by

$$c_n = \begin{cases} 0 & n = 3k \\ \frac{1}{n} & n = 3k + 1 \\ -\frac{1}{2n} & n = 3k + 2; \end{cases}$$

- (d) $(d_n) = (n - 10^{100}\sqrt{n})$.

Solution.

- (a) By the Archimedean Principle, for all $r \in \mathbf{R}$ there exists $n \in \mathbf{N}$ such that $n > r^2$. Hence $\sqrt{n} > r$. Therefore, the sequence (\sqrt{n}) is unbounded and diverges.

- (b) This sequence has two subsequences that converge to different limits, hence diverges.

The subsequence $(b_0, b_3, b_6, \dots) = (1, 1, 1, \dots)$ converges to 1.

The subsequence $(b_1, b_4, b_7, \dots) = (1/(3k+1))$ converges to 0 (by the Algebra of Limits Theorem).

- (c) The Theorem tells us nothing because this sequence converges. What do you think the limit is? How could you prove it?
- (d) Note that $d_n = \sqrt{n}(\sqrt{n} - 10^{100})$. By the Archimedean Principle, for all $r \in \mathbf{R}$ there exists a natural number m such that $\sqrt{m} > 10^{100} + 1$. Hence for all $n > m$, $d_n > \sqrt{n}$. Applying the Archimedean Principle again, we see that for all $r \in \mathbf{R}$ there exists $n \in \mathbf{N}$ (even larger than the m previously found) such that

$$d_n = n - 10^{100}\sqrt{n} = \sqrt{n}(\sqrt{n} - 10^{100}) > r.$$

Therefore the sequence is unbounded and so diverges. □

Topics: Cauchy sequences and applications

6.5 (Set(s) of Sequences). Draw a Venn-style diagram that illustrates the set Seq of all sequences (of real numbers). On this diagram, mark:

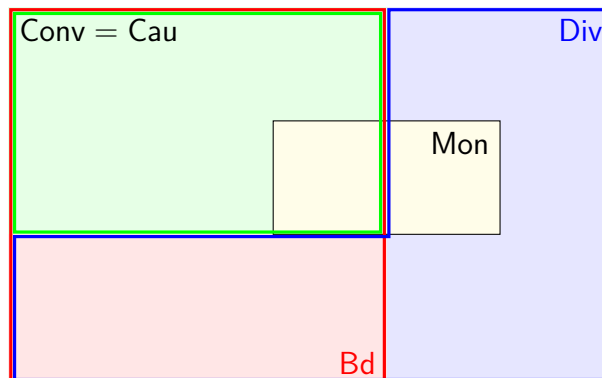
- the subset Bd of all bounded sequences;
- the subset Mon of all monotone sequences;
- the subset Cau of all Cauchy sequences;
- the subset Conv of all convergent sequences;
- the subset Div of all divergent sequences.

To do this accurately, you need to ask yourself: for any pair (A, B) of the above subsets, how are A and B related (e.g. are they disjoint? Is one a subset of the other? Are there elements in one that are not in the other?)

See if you can find explicit examples of sequences that belong to the various regions appearing in your diagram.

Solution. From results proved in the lectures we know that

$$\text{Bd} \cap \text{Mon} \subsetneq \text{Conv} = \text{Cau} \subsetneq \text{Bd} \quad \text{and} \quad \text{Conv} \cap \text{Div} = \emptyset.$$



$$\begin{aligned} ((-1)^n) \in \text{Bd} \setminus \text{Conv}, & \quad ((-1)^n/(n+1)) \in \text{Conv} \setminus \text{Mon}, & \quad (1/(n+1)) \in \text{Conv} \cap \text{Mon}, \\ (n) \in \text{Div} \cap \text{Mon}, & \quad ((-1)^n n) \in \text{Div} \setminus \text{Mon}. \end{aligned}$$

□

6.6 (A Recursive Sequence). Let (a_n) be the sequence defined by $a_0 = 0$ and

$$a_{n+1} = \sqrt{2 + a_n} \quad \text{for all } n \geq 0.$$

- Compute the first few terms of the sequence.
- Use induction to prove that $0 \leq a_n < 2$ for all $n \in \mathbf{N}$.
- You now know that (a_n) is bounded. What other property would you need to prove for (a_n) so that you can conclude that (a_n) converges?
- Use induction to prove the property you identified in the previous part, and therefore conclude that (a_n) converges.
- What is the limit of the sequence (a_n) ?

Solution.

(a) The first few terms are

$$\left(0, 2, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \dots \right)$$

This sequence seems to be increasing.

(b) Base case $n = 0$: we have $a_0 = 0 < 2$. And so $0 \leq a_0 < 2$.

Induction step: let $k \geq 0$ and assume $0 \leq a_k < 2$. We have

$$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2} = 2.$$

Further, by the definition of square roots, $\sqrt{2 + a_k} \geq 0$. Therefore $0 \leq a_{k+1} < 2$.

(c) We must show a_n is monotone.

(d) For $n = 1$, we have $a_{n-1} = 0$ and $a_n = \sqrt{2}$, therefore $a_n > a_{n-1}$.

Let $k \geq 1$ and assume $a_k > a_{k-1}$.

We have $a_{k+1} = \sqrt{2 + a_k}$ and $a_k = \sqrt{2 + a_{k-1}}$. Since $a_k > a_{k-1}$, we have $\sqrt{2 + a_k} > \sqrt{2 + a_{k-1}}$. Therefore, $a_{k+1} > a_k$.

Since (a_n) is bounded and monotone, it follows that (a_n) converges.

(e) Note that by definition

$$a_{n+1}^2 = 2 + a_n \quad \text{for all } n \in \mathbf{N}.$$

Take limits on both sides and use the Algebra of Limits Theorem to get that $L = 2$ or $L = -1$. Clearly it must be $L = 2$ since $0 \leq a_n < 2$ for all $n \in \mathbf{N}$. \square

6.7 (A Contractive Sequence). Let $(a_n)_{n \geq 1}$ be the sequence defined by $a_1 = 1$, $a_2 = 2$, and

$$a_n = \frac{a_{n-2} + a_{n-1}}{2} \quad \text{for all } n \geq 3.$$

(a) Compute the first few terms of the sequence.

(b) Use induction to prove that $1 \leq a_n \leq 2$ for all $n \geq 1$.

(c) Is the sequence monotone? Explain.

(d) Prove that for all $n \geq 1$ we have

$$|a_{n+2} - a_{n+1}| = \frac{1}{2} |a_{n+1} - a_n|.$$

(e) Conclude that the sequence converges. Can you find the limit?

Solution.

(a) $1, 2, \frac{3}{2}, \frac{7}{4}, \frac{13}{8}, \frac{27}{16}, \frac{53}{32}, \frac{107}{64}, \frac{213}{128}, \dots$

(b) Base cases $n = 1$ and $n = 2$: we have $a_1 = 1$ and $a_2 = 2$, so $1 \leq a_1, a_2 \leq 2$.

Induction step: let $k \geq 1$ and assume $1 \leq a_n \leq 2$ for all $n \leq k$. We have

$$1 = \frac{1}{2}(1 + 1) \leq \frac{1}{2}(a_{k-1} + a_k) \leq \frac{1}{2}(2 + 2) = 2.$$

Therefore $1 \leq a_{k+1} \leq 2$.

- (c) No. We know that $a_1 = 1$ and $a_2 = 2$ and we compute $a_3 = 3/2$. Thus $a_1 < a_2$ but $a_2 > a_3$. It follows that the sequence is neither monotone increasing nor monotone decreasing.

(Note that this behavior persists. It is not hard to prove by induction that for each integer $k \geq 1$, $a_{2k-1} < a_{2k}$ but $a_{2k} > a_{2k+1}$.)

- (d) We compute explicitly, for any $n \geq 1$:

$$|a_{n+2} - a_{n+1}| = \left| \frac{1}{2}(a_n + a_{n+1}) - a_{n+1} \right| = \left| \frac{1}{2}(a_{n+1} - a_n) \right| = \frac{1}{2}|a_{n+1} - a_n|.$$

- (e) We have shown that this sequence is **contractive** (with $c = 1/2$). Therefore, the sequence converges. We can try to use the Algebra of Limits Theorem to find the limit L , but all it will tell us is that $L = \frac{1}{2}(L + L)$, which is **True** but useless.

Side remark: in fact, the limit is $L = 5/3$.

This is beyond the scope of this subject, but we give an outline of the calculation here.

Rewrite the recursive definition of a_n as $2a_n - a_{n-1} - a_{n-2} = 0$, $a_1 = 1, a_2 = 2$.

Substituting $a_n = r^n$ gives $2r^n - r^{n-1} - r^{n-2} = 0$. Dividing by r^{n-2} yields the characteristic equation $2r^2 - r - 1 = 0$. Solving, $(2r + 1)(r - 1) = 0 \Rightarrow r = 1, r = -\frac{1}{2}$.

Since the roots of the characteristic equation are distinct, the general solution is $a_n = A \cdot 1^n + B \left(-\frac{1}{2}\right)^n = A + B \left(-\frac{1}{2}\right)^n$.

Use the initial conditions: $A + B \left(-\frac{1}{2}\right) = 1, A + B \left(\frac{1}{4}\right) = 2$ to get $B = \frac{4}{3}$ and $A = \frac{5}{3}$.

Thus $a_n = \frac{5}{3} + \frac{4}{3} \left(-\frac{1}{2}\right)^n$. Since $\left(-\frac{1}{2}\right)^n \rightarrow 0$, we conclude $a_n \rightarrow \frac{5}{3}$. \square

6.8 (Squares, Squares Everywhere). Consider the following statement:

- (*) For any $m, y \in \mathbf{Q}$ such that $0 < m^2 < y$, there exists $t \in \mathbf{Q}$ such that $m^2 < t^2 < y$.

- (a) Prove the statement. Aim for a short and simple proof, involving properties of the real numbers that we have seen.

[**Hint:** Use [Theorem 2.60](#).]

Read the statement (*) again and note that it does not involve the real numbers \mathbf{R} at all.

Is it possible to give a proof of statement (*) that does not involve \mathbf{R} at all, only properties of the rational numbers \mathbf{Q} ? Let's see.

- (b) Prove the Archimedean Property of \mathbf{N} in \mathbf{Q} : for every $x \in \mathbf{Q}$ with $x > 0$ there exists $n \in \mathbf{N}$ such that $n > x$.

- (c) Prove statement (*) without any reference to \mathbf{R} .

[**Hint:** Consider $t = m + \frac{1}{k}$, where $k \in \mathbf{N}$ is chosen such that $(2m + 1) < k(y - m^2)$.]

- (d) Is there a statement (*) that replaces squares by cubes, fourth powers, etc?

Write down a precise formulation of statement (*) for n -th powers (with $n \in \mathbf{N}, n \geq 1$).

Modify your proofs in parts (a) and (c) so that they work in this more general setting.

Solution.

- (a) Since $m, y \in \mathbf{Q}$ with $0 < m^2 < y$, we have $0 < m < \sqrt{y}$. Applying [Theorem 2.60](#) we get some $t \in \mathbf{Q}$ such that $0 < m < t < \sqrt{y}$. Therefore $0 < m^2 < t^2 < y$.

- (b) Let $x \in \mathbf{Q}$, $x > 0$. Write $x = a/b$ with $a, b \in \mathbf{N}$, $a, b \geq 1$. Since $b \geq 1$ we have $a = \frac{a}{1} \geq \frac{a}{b} = x$. Let $n = a + 1$, then $n > a \geq x$.
- (c) As the hint suggests, we use that Archimedean Property of \mathbf{N} in \mathbf{Q} to find $k \in \mathbf{N}$ such that

$$k > \frac{2m + 1}{y - m^2}.$$

(That $(2m + 1)/(y - x)$ is a positive rational number follows from the assumptions.)

Let $t = m + \frac{1}{k} \in \mathbf{Q}$. Then

$$0 < t^2 - m^2 = \frac{2m}{k} + \frac{1}{k^2} \leq \frac{2m}{k} + \frac{1}{k} = \frac{2m + 1}{k} < y - m^2,$$

so adding m^2 we get

$$m^2 < t^2 < y.$$

- (d) The statement is: For any $n \in \mathbf{N}$ with $n \geq 1$ and any $m, y \in \mathbf{Q}$ such that $0 < m^n < y$, there exists $t \in \mathbf{Q}$ such that $m^n < t^n < y$.

A simple proof using real numbers would be (modifying part (a)): $0 < m^n < y$ implies that $0 < m < y^{(1/n)}$, so applying [Theorem 2.60](#) we get $t \in \mathbf{Q}$ such that $0 < m < t < y^{(1/n)}$. Therefore $0 < m^n < t^n < y$.

For a proof using only rational numbers, we proceed in a manner similar to part (c).

Let $k \in \mathbf{N}$, $k \geq 1$. Starting with the Binomial Theorem we have

$$\begin{aligned} \left(m + \frac{1}{k}\right)^n &= \sum_{j=0}^n \binom{n}{j} \frac{m^{n-j}}{k^j} \\ &= m^n + \sum_{j=1}^n \binom{n}{j} \frac{m^{n-j}}{k^j} \\ &\leq m^n + \sum_{j=1}^n \binom{n}{j} \frac{m^{n-j}}{k} \\ &= m^n + \frac{1}{k} \sum_{j=1}^n \binom{n}{j} m^{n-j}. \end{aligned}$$

The inequality appearing above comes from $k \geq 1$, hence $k^j \geq k$ for all $j \geq 1$.

We now use the Archimedean Property of \mathbf{N} in \mathbf{Q} to find $k \in \mathbf{N}$ such that

$$k > \frac{1}{y - m^n} \sum_{j=1}^n \binom{n}{j} m^{n-j}.$$

(The fact that the right hand side of this inequality is a positive rational number follows from the assumptions on y and m , as well as the fact that the binomial coefficients are positive integers.)

Let $t = m + \frac{1}{k} \in \mathbf{Q}$, then by the computation we performed above we have

$$0 < t^n - m^n = \frac{1}{k} \sum_{j=1}^n \binom{n}{j} m^{n-j} < y - m^n,$$

so adding m^n we get

$$m^n < t^n < y. \quad \square$$