Topics: more sequences; limit points

7.1 (Zeno's Non-Paradox)**.** For $n \in \mathbf{N}$, let

$$x_n = \sum_{k=0}^{n} \frac{1}{2^k}.$$

(a) Write out the first few terms of the sequence (x_n).

(b) Prove that (x_n) is bounded and monotone.

(c) Prove that $\sup\{x_n : n \in \mathbf{N}\} = 2$.

(d) Find the limit of (x_n) as $n \longrightarrow \infty$.

Solution.

(a) $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots$

(b) $1 \leqslant x_n \leqslant 2$ for all $n \in \mathbf{N}$. That $1 \leqslant x_n$ is clear from $x_0 = 1$ and the fact that the sum has only positive terms. For the other inequality we observe that x_n is a geometric sum, hence $x_n = 2 - \frac{1}{2^n} \leqslant 2$.

That (x_n) is monotone increasing is clear from the fact that the sum has only positive terms.

(c) As we have seen above,

$$S = \{x_n : n \in \mathbf{N}\} = \left\{2 - \frac{1}{2^n} : n \in \mathbf{N}\right\},$$

and 2 is an upper bound for S.

Let $\varepsilon > 0$. By the Archimedean Principle, there exists $n \in \mathbf{N}$ such that $n > 1/\varepsilon$. Then $2^n \geqslant n > 1/\varepsilon$, so

$$\frac{1}{2^n} < \varepsilon \Rightarrow 2 - \frac{1}{2^n} > 2 - \varepsilon.$$

We conclude (by Theorem 2.50) that $\sup S = 2$.

(In the proof we used the fact that $2^n \geqslant n$ for all $n \geqslant 1$. Practice your mathematical induction skills by proving this fact.)

(d) Since (x_n) is monotone increasing, we know from Corollary 3.28 that its limit is equal to the supremum of its set of terms, hence $x_n \longrightarrow 2$. ☐

7.2 (Alternating Divergent)**.** Let (x_n) be a sequence converging to a limit $L \in \mathbf{R}$ and let (y_n) be the sequence given by $y_n = (-1)^n x_n$ for all $n \in \mathbf{N}$.

(a) Prove that if $L \neq 0$ then (y_n) diverges.

(b) What happens if $L = 0$? Make a precise statement and prove it.

Solution.

(a) By the Algebra of Limits, the subsequence $(y_{2k}) = (x_{2k})$ converges to L and the subsequence $(y_{2k+1}) = (-x_{2k+1})$ converges to $-L$. Since $L \neq 0$ we have $L \neq -L$, so (y_n) diverges.

(b) I claim that $(y_n) \longrightarrow 0$.

Note that $|y_n - 0| = |y_n| = |x_n| = |x_n - 0|$, so that the convergence of (x_n) to 0 is equivalent to the convergence of (y_n) to 0. ☐

7.3 (Limit Points of $\mathbf{N}$)**.** In Example 4.5 we have seen that if $a \in \mathbf{N}$, then a is not a limit point of $\mathbf{N} \subseteq \mathbf{R}$.

Now prove that if $a \in \mathbf{R} \setminus \mathbf{N}$, then a is not a limit point of $\mathbf{N} \subseteq \mathbf{R}$.

Solution. If $a < 0$, let $\delta = -a$. If $a > 0$, by the Archimedean Principle II there exists $n \in \mathbf{N}$ such that $n - 1 \leqslant a < n$; let $\delta = \mathsf{min}\{n - a, a - (n-1)\}$.

In either case we have $\delta > 0$ and δ is the distance to the nearest natural number, so there exists no $x \in \mathbf{N}$ such that $0 < |x - a| < \delta$.

Therefore a is not a limit point of $\mathbf{N}$. ☐

7.4 (Limit Points)**.** Find all limit points of each of the following subsets of $\mathbf{R}$:

(a) $(1, 2)$

(b) $(1, 2]$

(c) $\{0\}$

(d) $\mathbf{Q}$

(e) $(-\infty, -1) \cup (1, \infty)$

(f) $\mathbf{R} \setminus \mathbf{N}$

(g) $\{1/n : \ n \in \mathbf{N}\}$.

Solution.

(a) $[1, 2]$

(b) $[1, 2]$

(c) $\varnothing$

(d) $\mathbf{R}$

(e) $(-\infty, -1] \cup [1, \infty)$

(f) $\mathbf{R}$

(g) $\{0\}$. ☐

7.5 (Limits from Scratch)**.** Using the $\varepsilon - \delta$ definition of limit of a function, prove that each of the statements below is True.

(a) $\lim\limits_{x \to 2} 1 = 1$

(b) $\lim\limits_{x \to a} 1 = 1$

(c) $\lim\limits_{x \to 2} x = 2$

(d) $\lim\limits_{x \to a} x = a$

(e) $\lim_{x\to 2} x^2 = 4$

(f) $\lim_{x\to a} x^2 = a^2$ for $a > 0$.

Solution.

(a) Any δ works. For example, take $\delta = 1$.

(b) Again, any δ works. For example, take $\delta = 1$.

(c) Let $\delta = \varepsilon$.

(d) Let $\delta = \varepsilon$.

(e) Let $\varepsilon > 0$. Let $\delta = \mathsf{min}\{1, \varepsilon/5\}$. First suppose that $\varepsilon \geqslant 5$ so that $\delta = 1$. If $|x - 2| < 1$, then $1 < x < 3$ so $1 < x^2 < 9$. It follows that $|x^2 - 4| < 5 = \varepsilon$.

Now suppose that $\varepsilon < 5$ so that $\delta = \varepsilon/5$. Since $|x - 2| < 1$, by the triangle inequality

$$|x + 2| = |(x + 2) - 4| \leqslant |x - 2| + 4 \leqslant 5.$$

Therefore

$$|x^2 - 4| = |x + 2|\,|x - 2| \leqslant 5\,|x - 2| < 5\delta = \varepsilon.$$

(f) Let $\delta = \mathsf{min}\{1, \varepsilon/(1 + 2a)\}$. Proceed as in (e), swapping 4 for a^2. ☐

Topics: limits of functions, one-sided limits, sequential methods

7.6 (One-Sided Limits)**.** Let $E \subseteq \mathbf{R}$, let $f : E \longrightarrow \mathbf{R}$ be a function, and let $a, L \in \mathbf{R}$. Suppose that a is a limit point of the set $E \cap (a, \infty)$.
We write

$$\lim_{x \to a^+} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.

(a) Write down the analogous definition of $\lim_{x \to a^-} f(x) = L$.

(b) Consider the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 2x & x > 1, \\ -2x & x < 1, \\ 0 & x = 1. \end{cases}$$

Sketch a graph of the function.

Prove that

$$\lim_{x \to 1^-} f(x) = -2 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 2.$$

(c) Back to the general setting.

Let $E \subseteq \mathbf{R}$, $f : E \longrightarrow \mathbf{R}$ a function, and $L \in \mathbf{R}$. Suppose that $a \in \mathbf{R}$ is a limit point of both sets $E \cap (-\infty, a)$ and $E \cap (a, \infty)$. Prove that

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

(d) What can you say about the limit of the function f from part (b) as $x \longrightarrow 1$?

Solution.

(a) Suppose that a is a limit point of the set $E \cap (-\infty, a)$.

We write

$$\lim_{x \to a^-} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$.

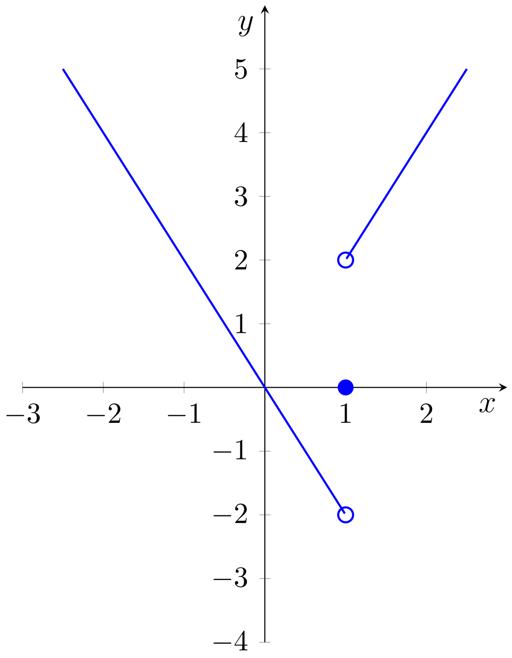
(b) Let $\varepsilon > 0$. Set $\delta = \varepsilon/2$. If $x \in \mathbf{R}$ is such that $1 < x < 1 + \delta$, then

$$|f(x) - 2| = |2x - 2| = 2|x - 1| < 2\delta = \varepsilon.$$

Therefore $\lim_{x \to 1^+} f(x) = 2$.

The other one-sided limit is proved similarly.

Here is the graph:



(c) It follows directly from the definitions that if $\lim_{x \to a} f(x) = L$ then $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

Let's consider the other direction. Suppose $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

Let $\varepsilon > 0$. There exists $\delta_- > 0$ such that if $x \in E$ and $a - \delta_- < x < a$, then $|f(x) - L| < \varepsilon$. There exists $\delta_+ > 0$ such that if $x \in E$ and $a < x < a + \delta_+$, then $|f(x) - L| < \varepsilon$.

Let $\delta = \mathsf{min}\{\delta_-, \delta_+\}$. Suppose $x \in E$ and $0 < |x - a| < \delta$. Then $x \neq a$ and

$$a - \delta_- \leqslant a - \delta < x < a + \delta \leqslant a + \delta_+.$$

If $x < a$, then $a - \delta_- < x < a$, so $|f(x) - L| < \varepsilon$. If $x > a$, then $a < x < a + \delta_+$, so $|f(x) - L| < \varepsilon$.

In all possible cases we conclude that $|f(x) - L| < \varepsilon$, so $\lim_{x \to a} f(x) = L$.

(d) By part (c), the limit of the function does not exist since the two one-sided limits at 1 are $-2 \neq 2$. ☐

7.7 (Inequalities and Limits of Functions)**.** Prove Theorem 4.12:

"Let $E \subseteq \mathbf{R}$, a a limit point of E, and $f, g :\ E \longrightarrow \mathbf{R}$ such that

$$\lim_{x \to a} f(x) = \alpha \quad \text{and} \quad \lim_{x \to a} g(x) = \beta.$$

If $f(x) \leqslant g(x)$ for all $x \in E$, then $\alpha \leqslant \beta$."

[**Hint**: Combine Theorem 4.10 and Theorem 3.16.]

Solution. By the Sequential Criterion for Function Limits (Theorem 4.10), for every sequence (a_n) such that $a_n \in E \setminus \{a\}$ for all $n \in \mathbf{N}$ and $a_n \longrightarrow a$, we have that $f(a_n) \longrightarrow \alpha$ and $g(a_n) \longrightarrow \beta$.

But $f(a_n) \leqslant g(a_n)$ for all $n \in \mathbf{N}$, so by the Inequalities and Limits of Sequences Theorem 3.16 we have $\alpha \leqslant \beta$. ☐

7.8. Prove the Sandwich Theorem for Functions, which is the following statement:

Let $E \subseteq \mathbf{R}$, let a be a limit point of E, let $f, g, h : E \longrightarrow \mathbf{R}$, and $L \in \mathbf{R}$. Suppose

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

and

$$f(x) \leqslant g(x) \leqslant h(x) \quad \text{for all } x \in E.$$

Then

$$\lim_{x \to a} g(x) = L.$$

Solution. We'll use the Sequential Criterion for Function Limits and the Sandwich Theorem for Sequences (Exercise 3.7).

Let (a_n) be a sequence such that $a_n \in E \setminus \{a\}$ for all $n \in \mathbf{N}$ and $a_n \longrightarrow a$. Then $f(a_n) \longrightarrow L$ and $h(a_n) \longrightarrow L$. But $f(a_n) \leqslant g(a_n) \leqslant h(a_n)$ for all $n \in \mathbf{N}$, so by the Sandwich Theorem for Sequences we get that $g(a_n) \longrightarrow L$.

Therefore, using the Sequential Criterion again, we conclude that $\lim_{x \to a} g(x) = L$. ☐

7.9. Using the Sandwich Theorem, find

$$\lim_{x \to 0} x^2 \sin(x).$$

Solution. We have for all $x \in \mathbf{R}$:

$$-1 \leqslant \sin(x) \leqslant 1 \Rightarrow -x^2 \leqslant x^2 \sin(x) \leqslant x^2.$$

By Tutorial Question 7.5 (or the Algebra of Limits) we have

$$\lim_{x \to 0} x^2 = 0 = \lim_{x \to 0} (-x^2),$$

so by the Sandwich Theorem:

$$\lim_{x \to 0} x^2 \sin(x) = 0.$$

☐

7.10 (Characteristic Function of $\mathbf{Q}$)**.** Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$g(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q}. \end{cases}$$

Let $a \in \mathbf{R}$.

(a) Prove that there exists a sequence (a_n) with $a_n \in \mathbf{Q}$ for all $n \in \mathbf{N}$ and $a_n \longrightarrow a$.

[**Hint**: Use the fact that $\mathbf{Q}$ is dense in $\mathbf{R}$, see Corollary 2.61.]

(b) It is also the case that $\mathbf{R} \setminus \mathbf{Q}$ is dense in $\mathbf{R}$. Use this to prove that there exists a sequence (b_n) with $b_n \in \mathbf{R} \setminus \mathbf{Q}$ for all $n \in \mathbf{N}$ and $b_n \longrightarrow a$.

(c) Find $\lim_{x \to a} g(x)$ (or show that it does not exist).

Solution.

(a) Fix $n \in \mathbf{N}$ and let $\varepsilon = 1/(n+1)$. By Corollary 2.61, there exists $a_n \in \mathbf{Q}$ such that $|a - a_n| < \varepsilon$, in other words

$$a - \frac{1}{n+1} < a_n < a + \frac{1}{n+1} \qquad \text{for all } n \in \mathbf{N}.$$

Using the Sandwich Theorem for sequences, we conclude that $a_n \longrightarrow a$.

(b) Use the exact same argument as in (a), but taking $b_n \in \mathbf{R} \setminus \mathbf{Q}$.

(c) We prove by contradiction that the limit does not exist. Suppose $L \in \mathbf{R}$ is such that $\lim_{x \to a} g(x) = L$. Consider a sequence (a_n) as in part (a). By Theorem 4.10, $\lim_{n \to \infty} g(a_n) = L$, but $g(a_n) = 1$ for all $n \in \mathbf{N}$, so $L = 1$.

Now take a sequence (b_n) as in part (b). We have $\lim_{n \to \infty} g(b_n) = L$, but $g(b_n) = 0$ for all $n \in \mathbf{N}$, so $L = 0$, contradiction. $\square$