Topics: definition and properties of continuity

8.1 (Proving and Disproving Continuity).

(a) Using the definition of continuity, prove that $f:\ \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 3x + 1$ is continuous at 0.

(b) Fix $k \in \mathbf{R}$. Using the definition of continuity, prove that the constant function $f:\ \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = k$ is continuous.

(c) Let $f, g:\ \mathbf{R} \longrightarrow \mathbf{R}$ be continuous at c. Using the definition of continuity, prove that $2f + 2g$ is continuous at c.

(d) Carefully write the negation of the definition of continuity: "The function $f:\ E \longrightarrow \mathbf{R}$ is not continuous at $a \in E$ if ..."

(e) Using part (d), show that the following function $f:\ \mathbf{R} \longrightarrow \mathbf{R}$ is not continuous at 0:

$$f(x) = \begin{cases} |x| & x \neq 0, \\ 1 & x = 0. \end{cases}$$

Solution.

(a) Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{3}$. Let x satisfy $|x - 0| < \delta$.

$$\begin{aligned} |f(x) - f(0)| &= |3x + 1 - 1| \\ &= |3x| \\ &= 3|x| \\ &< 3\delta \\ &= \varepsilon \end{aligned}$$

Therefore $f(x)$ is continuous at 0.

(b) Let $\varepsilon > 0$, and let $\delta = \varepsilon$, for any $x \in \mathbf{R}$, such that $|x - a| < \delta$, we have

$$|f(x) - f(a)| = |k - k| = 0 < \varepsilon.$$

Therefore, f is continuous at a. Since a was arbitrary, f is continuous on $\mathbf{R}$.

(c) Let $\varepsilon > 0$ and let $\varepsilon' = \frac{\varepsilon}{4}$. Since f is continuous at c there exists $\delta_f > 0$ so that $|f(x) - f(c)| < \varepsilon'$ whenever $|x - c| < \delta_f$. Analogously, there exists $\delta_g > 0$.

Let $\delta = \min\{\delta_f, \delta_g\}$. If $|x - c| < \delta$, then by the Triangle Inequality we have

$$|2f(x) + 2g(x) - 2f(c) - 2g(c)| \leqslant 2|f(x) - f(c)| + 2|g(x) - g(c)| < 2\varepsilon' + 2\varepsilon' = \varepsilon.$$

(d) "...there exists $\varepsilon > 0$ such that, for all $\delta > 0$, there exists $x \in E$ with $|x - a| < \delta$ and $|f(x) - f(a)| \geqslant \varepsilon$."

(e) Take $\varepsilon = \frac{1}{2}$, and let $\delta > 0$. Take $x = \min\{\frac{\delta}{2}, \frac{1}{4}\}$. Then $x \neq 0$ because $\delta > 0$, and we have $f(x) = \min\{\delta/2, 1/4\} \leqslant 1/4$. Therefore,

$$|f(x) - f(0)| = |f(x) - 1| = 1 - f(x) \geqslant \frac{3}{4} > \frac{1}{2}.$$

Thus, we have found an $\varepsilon > 0$ such that, for all $\delta > 0$, there is x with $|x - 0| < \delta$ and $|f(x) - f(0)| \geqslant \varepsilon$. By part (d), this means that $f(x)$ is not continuous at 0. □

8.2 (Continuity and Local Boundedness)**.** Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be continuous at c. Using the definition of continuity, prove the following facts:

(a) There exists $\delta_+ > 0$ so that $f(x)$ is bounded above by $f(c) + \frac{1}{2}$ on $(c - \delta_+, c + \delta_+)$.

(b) There exists $\delta_- > 0$ so that $f(x)$ is bounded below by $f(c) - \frac{1}{2}$ on $(c - \delta_-, c + \delta_-)$.

(c) For every $\varepsilon > 0$ there exists $\delta > 0$ so that

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon \qquad \text{for all } x \in (c - \delta, c + \delta).$$

Solution. For (a) and (b), let $\varepsilon = 1/2$. Since f is continuous at c, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{1}{2} \qquad \text{for all } x \text{ such that } |x - c| < \delta,$$

which can be rewritten as

$$f(c) - \frac{1}{2} < f(x) < f(c) + \frac{1}{2} \qquad \text{for all } x \in (c - \delta, c + \delta).$$

So we can take $\delta_+ = \delta_- = \delta$.

Part (c) is just the definition of continuity at c. □

8.3 (Sequences are Continuous)**.**

(a) Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$, and $a \in E$. Prove that if a is **not** a limit point of E, then f is continuous at a.

From now on, fix a sequence (f_n), and think of it as a function $f: \mathbf{N} \longrightarrow \mathbf{R}$ given by $f(n) = f_n$.

(b) What are the limit points of $\mathbf{N}$?

(c) Is f a continuous function on $\mathbf{N}$?

(d) Consider the statement: "every sequence is a continuous function $\mathbf{N} \longrightarrow \mathbf{R}$". Based on the work you have done above, is this statement True?

Solution.

(a) Suppose a is not a limit point of E.

Let $\varepsilon > 0$. Since a is not a limit point of E, there exists $\delta > 0$ such that if $x \in E$ and $|x - a| < \delta$ then $x = a$.

Then there are no $x \in E$ such that $0 < |x - a| < \delta$, therefore the continuity condition is vacuously True.

(b) We have seen in Example 4.5 and Tutorial Question 7.3 that $\mathbf{N}$ has no limit points.

(c) Let $a \in \mathbf{N}$. Since $\mathbf{N}$ has no limit points (part (b)), a is not a limit point of $\mathbf{N}$, so f is continuous at a (part (a)).

Hence f is a continuous function on $\mathbf{N}$.

(d) Yes! □

8.4 (Restriction and Continuity)**.** Let $E \subseteq \mathbf{R}$ and let $f: E \longrightarrow \mathbf{R}$.

Given a subset $A \subseteq E$, recall that the *restriction of f to A* is the function $f|_A : A \longrightarrow \mathbf{R}$ given by

$$f|_A(x) = f(x) \qquad \text{for all } x \in A.$$

Prove that if $a \in A$ and f is continuous at a, then $f|_A$ is continuous at a.

Solution. Let $\varepsilon > 0$. Since f is continuous at a, there exists $\delta > 0$ such that if $x \in E$ with $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

I claim that the same $\delta > 0$ works for the restriction $f|_A$. Let $x \in A$ be such that $|x - a| < \delta$. As $A \subset E$, we have that $x \in E$ is such that $|x - a| < \delta$, so

$$\big|f|_A(x) - f|_A(a)\big| = |f(x) - f(a)| < \varepsilon. \qquad \square$$

Topics: extreme and intermediate value theorems

Refresh your memory on the precise statements of the Extreme Value Theorem and the Intermediate Value Theorem.

8.5 ("Counterexamples")**.**

(a) Give an example of a continuous function $f:\ E \longrightarrow \mathbf{R}$ such that: for every $x \in E$ there exists $x' \in E$ such that $f(x) < f(x')$.

(b) Why does the Extreme Value Theorem not apply to your function from (a)?

(c) Give an example of a function $f:\ E \longrightarrow \mathbf{R}$ such that: there exist $a, b \in E$ and $y \in \mathbf{R}$ such that
$$f(a) < y < f(b),$$
but for all $c \in [a, b]$ we have $f(c) \neq y$.

(d) Why does the Intermediate Value Theorem not apply to your function from (c)?

Solution.

(a) Take $f:\ (0, 1] \longrightarrow \mathbf{R}$ given by $f(x) = 1/x$. Given $x \in (0, 1]$, let $x' = x/2 \in (0, 1]$, then
$$f(x) = \frac{1}{x} < \frac{2}{x} = \frac{1}{x'} = f(x').$$

(b) The Extreme Value Theorem does not apply because the domain $E = (0, 1]$ of f is not a closed interval.

(c) Take $f:\ [1, 3] \longrightarrow \mathbf{R}$ given by
$$f(x) = \begin{cases} x & \text{if } x \neq 2, \\ 0 & \text{if } x = 2. \end{cases}$$
Let $y = 2$. Then, for any $c \in [1, 3]$ we have $f(c) \neq 2 = y$.

(d) The Intermediate Value Theorem does not hold here because f is not continuous on the interval $[1, 3]$. □

8.6. Let $h:\ [-2, 2] \longrightarrow \mathbf{R}$ be given by $h(x) = x^3 - 2x^2 + 1$. Let S be defined as in the proof of Lemma 4.33 to the Intermediate Value Theorem, namely
$$S = \big\{x \in [-2, 2] :\ h(t) \leqslant 0 \text{ for all } t \in [-2, x]\big\}.$$
What is $\sup S$?

[**Hint**: Find all the solutions of the equation $h(x) = 0$.]

Solution. The supremum of S is the smallest $x \in [-2, 2]$ such that $h(x) = 0$.

Some experimentation yields $h(1) = 0$, after which a bit of algebra gives
$$h(x) = \big(x^2 - x - 1\big)\big(x - 1\big),$$
so the solutions of $h(x) = 0$ are (in increasing order):
$$\frac{1 - \sqrt{5}}{2}, 1, \frac{1 + \sqrt{5}}{2}.$$
We conclude that
$$\sup S = \frac{1 - \sqrt{5}}{2}.$$
□

8.7. Use the definition of continuity to prove that: if $E \subseteq \mathbf{R}$, c is a limit point of E, $f: E \longrightarrow \mathbf{R}$ is continuous at c, and $f(c) < 0$, then there exists $\delta > 0$ such that

$$f(x) < 0 \qquad \text{for all } x \in (c - \delta, c + \delta).$$

Solution. Take $\varepsilon = -f(c)$. Since $f(c) < 0$, we have $\varepsilon > 0$.

Since f is continuous at c, there exists $\delta > 0$ such that if $x \in E$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. In other words:

$$f(x) < \varepsilon + f(c) = -f(c) + f(c) = 0.$$

□

8.8. Use the Intermediate Value Theorem in the following. Make sure to check that the hypotheses of the Theorem hold before applying it. (You may use without proof the fact that $\sin x$ is continuous on $\mathbf{R}$ and $\sqrt{x}$ is continuous on $[0, \infty)$.)

(a) Prove that the equation $x^2 = 12 + \sqrt{x}$ has at least one real solution.

(b) Prove that the equation $2 \sin x = x$ has at least three real solutions.

(c) Prove that the equation $x^3 = 2x^2 + 3x - 2$ has at least three real solutions.

Solution.

(a) $f(x) = x^2 - 12 - \sqrt{x}$ is a continuous function on $[3, 4]$.

Notice $f(3) < 0 < f(4)$. Therefore, by the Intermediate Value Theorem, there is a solution $f(x) = 0$ for some $x \in (3, 4)$.

(b) $f(x) = \sin x - \frac{1}{2}x$ is a continuous function on $\mathbf{R}$.

Notice $f(\pi/2) > 0 > f(\pi)$. Therefore, by the Intermediate Value Theorem, there is a solution $f(x) = 0$ for $x \in (\pi/2, \pi)$.

Similarly, there is a solution in $x \in (-\pi, -\pi/2)$. Also $x = 0$ is a solution.

(c) $f(x) = x^3 - 2x^2 - 3x + 2$ is a continuous function on $\mathbf{R}$.

Notice

$$f(-2) = -8 < 0, \quad f(0) = 2 > 0, \quad f(1) = -2 < 0, \quad f(3) = 2 > 0.$$

Applying the Intermediate Value Theorem, there exist $c_1 \in (-2, 0)$, $c_2 \in (0, 1)$, and $c_3 \in (1, 3)$ such that $f(c_1) = f(c_2) = f(c_3) = 0$. □

8.9. Use the Intermediate Value Theorem in the following. Make sure to check that the hypotheses of the Theorem hold before applying it.

(a) Let $f, g: [a, b] \longrightarrow \mathbf{R}$ be continuous such that

$$f(a) < g(b) < g(a) < f(b).$$

Prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

(b) Let $f: [0, 1] \longrightarrow [0, 1]$ be continuous. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$. (In other words, f has at least one *fixed point* on $[0, 1]$.)

(c) Can you find explicit functions f as in part (b) that have: exactly one fixed point? infinitely many fixed points? exactly three fixed points?

[**Hint**: Draw some pictures first, then see if you can find functions that match your pictures.]

Solution.

(a) Let $h(x) = f(x) - g(x)$. This function is continuous by the Algebra of Continuity Theorem. Notice $h(x) = 0$ precisely when $f(x) = g(x)$. By hypothesis, $h(a) < 0$ and $h(b) > 0$. Hence by the Intermediate Value Theorem, there exists $c \in (a, b)$ such that $h(c) = 0$.

(b) Consider $g(x) = f(x) - x$, which is continuous by the Algebra of Continuity Theorem. If $f(0) = 0$ or $f(1) = 1$, then we are done. Otherwise, since the codomain of f is $[0, 1]$, $f(0) > 0$ and $f(1) < 1$ so that $g(0) > 0$ and $g(1) < 0$. By the Intermediate Value Theorem, there exists $c \in (a, b)$ such that $g(c) = 0$; that is, $f(c) = c$.

(c) Exactly one fixed point: $f_1(x) = 1 - x$ has unique fixed point $c = 1/2$.

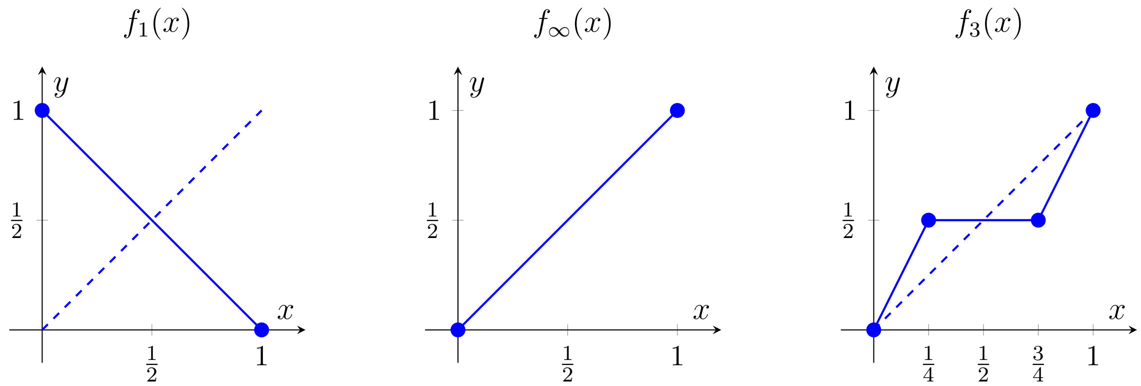
Infinitely many fixed points: $f_\infty(x) = x$ has every $c \in [0, 1]$ as fixed point.

Exactly three fixed points:

$$f_3(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ \frac{1}{2} & \text{if } \frac{1}{4} \leqslant x \leqslant \frac{3}{4}, \\ 2x - 1 & \text{if } \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$

This has $c = 0, 1/2, 1$ as fixed points.

This is what these three functions look like:



$\square$