

Topics: pointwise and uniform convergence

9.1 (Pointwise convergence). For each of the following sequences of functions $f_n : \mathbf{R} \rightarrow \mathbf{R}$, decide if it converges pointwise on the domain \mathbf{R} to some function f , and if so, find f .

- (a) $f_n(x) = \frac{x^2}{n}$ for $n \geq 1$;
- (b) $f_n(x) = \sin(nx^2)$ for $n \geq 0$;
- (c) $f_n(x) = \frac{1}{nx^2+1}$ for $n \geq 0$.

Solution.

- (a) This sequence converges pointwise to the constant function $f(x) = 0$.

To see this, fix $x \in \mathbf{R}$ and apply the Algebra of Limits to the sequence (x^2/n) :

$$\lim_{n \rightarrow \infty} \frac{x^2}{n} = (x^2) \lim_{n \rightarrow \infty} \frac{1}{n} = x^2 \cdot 0 = 0.$$

- (b) This sequence does not converge pointwise on \mathbf{R} . For example, if $x = \sqrt{\pi/2}$, then we have the sequence $\sin(n\pi/2) = (-1)^n$, which diverges.
- (c) This sequence converges pointwise to the function

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

First fix $x = 0$, then $f_n(0) = 1$ for all $n \geq 0$, so $f_n(0) \rightarrow 1$ as $n \rightarrow \infty$.

Now fix $x \neq 0$ and apply the Algebra of Limits:

$$\lim_{n \rightarrow \infty} \frac{1}{nx^2 + 1} = \lim_{n \rightarrow \infty} \frac{(1/n)}{x^2 + (1/n)} = \frac{0}{x^2 + 0} = 0. \quad \square$$

9.2 (Pointwise but not uniform). The objective is to prove, using the definitions, that the sequence of functions $f_n : [0, 1] \rightarrow \mathbf{R}$ defined by $f_n(x) = x^n$ converges pointwise, but not uniformly, to the function

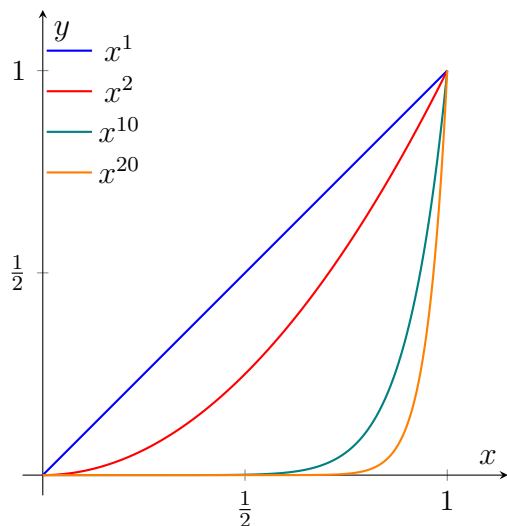
$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

- (a) Using any graphing calculator or system (e.g. Wolfram Alpha is available online and will work), graph some example functions $y = x^n$, $0 \leq x \leq 1$. (Say, for $n = 1, 2, 10, 20$.) What do you notice?
- (b) Now we prove the sequence converges pointwise:
 - i. Show that the sequence (x^n) is monotone decreasing for all $0 \leq x < 1$.
 - ii. Show that, for each $0 \leq x < 1$, 0 is the greatest lower bound of $\{x^n | n \in \mathbf{N}\}$.
 - iii. Using the above, show that $f_n \rightarrow f$ pointwise on $[0, 1]$.
- (c) Finally, we prove that the convergence is not uniform:
 - i. Write out the statement “ (f_n) does not converge uniformly to f ” in formal mathematical language.

- ii. Show that, for every $n \in \mathbf{N}$, there exists a real number $c \in (0, 1)$ such that $c^n \geq 1/2$.
- iii. Now prove that the sequence (f_n) does not converge uniformly to f .

Solution.

- (a) As n grows, the graph more and more resembles a horizontal line along the x -axis until we get very close to 1, and then grows to 1 very rapidly:



- (b)
 - i. Since $x < 1$, $x^{n+1} = x \cdot x^n < 1 \cdot x^n = x^n$ for all $n \geq 1$. Thus, the sequence is monotone decreasing.
 - ii. Since $x \geq 0$, all terms x^n are non-negative, so 0 is a lower bound. If $x = 0$, then this is the greatest lower bound by inspection. Now suppose $0 < x < 1$, and assume for a contradiction that $\alpha > 0$ is the infimum of the set $\{x^n | n \in \mathbf{N}\}$. Then $\alpha \leq x < 1$, and therefore $\alpha < \sqrt{\alpha}$. By definition of infimum, there exists $N \in \mathbf{N}$ such that $x^N < \sqrt{\alpha}$. But then $x^{2N} < x^N \sqrt{\alpha} < \alpha$, a contradiction to the hypothesis that α is a lower bound. We have proved that 0 is the greatest lower bound.
 - iii. It follows from the above and the monotone convergence theorem that $x^n \rightarrow 0$ when $0 \leq x < 1$. When $x = 1$, the sequence (x^n) is the constant sequence (1), which converges to 1. Thus, $f_n \rightarrow f$ pointwise on $[0, 1]$.
- (c)
 - i. $(\exists \varepsilon > 0) (\forall M \in \mathbf{N}) (\exists x \in [0, 1]) (\exists n \in \mathbf{N}) [n > M \wedge |f_n(x) - f(x)| \geq \varepsilon]$.
 - ii. Take $c = \sqrt[n]{1/2}$.
 - iii. Choose $\varepsilon = 1/2$. Given M , let $n = M + 1$ and choose for x the value obtained in part (ii). □

9.3 (Uniformly, but where?). Uniform convergence is a property of a sequence of functions on a given domain. We explore what this means in an example.

- (a) Carefully write a proof that the sequence of functions $f_n(x) = x/n$ converges pointwise to 0 on the domain \mathbf{R} . Pay attention to your choice of M and how it depends on x .
- (b) Show that the sequence $f_n(x) = x/n$ **does not** converge uniformly on the domain \mathbf{R} .
- (c) Show that, in contrast, the sequence $f_n(x) = x/n$ **does** converge uniformly on the domain $[-1, 1]$.

- (d) What happens if you replace $f_n(x) = x/n$ by $g_n(x) = \frac{1}{nx^2+1}$ above? Is part (c) still True? Briefly explain.
- (e) Complete the following statement, and prove it:
 Let $E \subseteq \mathbf{R}$. Then the sequence of functions $f_n(x) = x/n$ converges uniformly on E if and only if [...].

Solution.

- (a) Given $x \in \mathbf{R}$ and $\varepsilon > 0$, choose $M = \lceil |x|/\varepsilon \rceil$ in the ε - M proof (except when $x = 0$, when we choose $M = 1$.) Since we obtain this M by solving an inequality, it is indeed the smallest possible choice.
- (b) We can take $\varepsilon = 1$. Given $M \in \mathbf{N}$, let $n = M + 1$. Then, for any $x \geq n$, $|x/n - 0| \geq 1$. Thus, the sequence does not converge uniformly.
- (c) From part (a), we see that we can choose $M = \lceil 1/\varepsilon \rceil$, and this will work for all x such that $|x| \leq 1$. Thus, given $\varepsilon > 0$, we can choose M that does not depend on $x \in [-1, 1]$. By definition, this means that $x/n \rightarrow 0$ uniformly in the domain $[-1, 1]$.
- (d) No, because the limit function (found in [Tutorial Question 9.1](#)) is not continuous.
- (e) Let $E \subseteq \mathbf{R}$. Then the sequence of functions $f_n(x) = x/n$ converges uniformly on E if and only if E is bounded.

Here is a proof.

Suppose first that $x/n \rightarrow 0$ uniformly on E . Then there exists $M \in \mathbf{N}$ such that, for all $x \in E$ and for all $n > M$, $|x/n| < 1$. In particular, $|x/(M + 1)| < 1$, and therefore, $|x| < M + 1$. This shows that E is bounded.

Now suppose E is bounded. By definition, this means that there exists $R > 0$ such that, for all $x \in E$, $|x| < R$. Let $\varepsilon > 0$. Set $M = \lceil R/\varepsilon \rceil$. If $n > M$, then

$$\left| \frac{x}{n} \right| \leq \frac{R}{n} < \frac{R}{M} \leq \varepsilon.$$

Thus, $x/n \rightarrow 0$ uniformly on the domain E . □

Topics: definition and properties of derivatives

9.4 (A differentiable piecewise function). Let $k \in \mathbf{R}$, and let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$f(x) = \begin{cases} kx - k + 1 & x < 1, \\ x^3 & x \geq 1. \end{cases}$$

Determine all values of k for which f is differentiable at $c = 1$.

Solution. Noting that $f(1) = 1$, we compute that

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = k \quad \text{and} \quad \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = 3.$$

Therefore f is differentiable at $c = 1$ if and only if $k = 3$. □

9.5 (Derivatives of increasing functions). Let $f : [a, b] \rightarrow \mathbf{R}$ be a function.

We say that f is *monotone increasing on $[a, b]$* if for all $x, z \in [a, b]$ with $x \leq z$ we have $f(x) \leq f(z)$.

Prove that if f is differentiable and monotone increasing on $[a, b]$, then $f'(c) \geq 0$ for all $c \in [a, b]$.

Solution. Since f is differentiable on $[a, b]$, for all $c \in [a, b]$ there exists $L \in \mathbf{R}$ so that

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = L.$$

Examining the left and right side limits, for all $c \in (a, b)$ we have

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} = L = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}.$$

Approaching from the right, we have $x > c$. Since f is monotone increasing, we have $f(x) \geq f(c)$. Therefore

$$\frac{f(x) - f(c)}{x - c} \geq 0.$$

Therefore

$$L = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0.$$

By a similar argument, $f'(a) \geq 0$ and $f'(b) \geq 0$. Therefore $f'(c) \geq 0$ for all $c \in [a, b]$. □

9.6 (Locally non-constant). Let $f : [a, b] \rightarrow \mathbf{R}$ be differentiable on $[a, b]$ and let $c \in (a, b)$.

- (a) Prove that if $f'(c) \neq 0$, then there exists $\delta > 0$ so that $f(x) \neq f(c)$ for all $x \in (c - \delta, c + \delta)$ with $x \neq c$.
- (b) Is the converse of the statement in (a) True?

Solution.

- (a) We prove the contrapositive. Assume for all $\delta > 0$ the statement

$$f(x) \neq f(c) \text{ for all } x \in (c - \delta, c + \delta) \text{ with } x \neq c.$$

is **False**. Therefore for all $\delta > 0$ there exists $x \in (c - \delta, c + \delta)$ so that $x \neq c$ and $f(x) = f(c)$. We prove $f'(c) = 0$.

Since f is differentiable on $[a, b]$, there exists $L \in \mathbf{R}$ so that

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = L.$$

Therefore for every $\varepsilon > 0$ there exists $\delta > 0$ so that $\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$ whenever $0 < |x - c| < \delta$. By hypothesis, for every such δ there exists $z \in (c - \delta, c + \delta)$ so that $f(z) = f(c)$. And so, for every $\varepsilon > 0$ there exists $z \in (c - \delta, c + \delta)$ so that

$$\varepsilon > \left| \frac{f(z) - f(c)}{z - c} - L \right| = | -L | \geq 0.$$

Therefore we have

$$0 \leq | -L | < \varepsilon$$

for every $\varepsilon > 0$. And so we conclude $L = 0$.

- (b) No. Consider the function $f(x) = x^3$. This function is strictly increasing, so for all $x \neq 0$, $f(x) \neq f(0) = 0$. But, $f'(0) = 0$. □

9.7 (A discontinuous derivative). Consider the function $f : \mathbf{R} \rightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

- (a) Determine whether f is continuous on \mathbf{R} .
 (b) Find the derivative at all points $c \neq 0$.
 (c) Find the derivative at $c = 0$. Appeal to left and right side limits.
 (d) Determine whether $f'(x)$ is continuous at 0.

Solution.

- (a) Yes. To show this, you can use the fact that $|\sin(1/x)| \leq 1$ for $x \neq 0$, and hence

$$-x^2 \leq f(x) \leq x^2.$$

Since $\lim_{x \rightarrow 0} x^2 = 0$, the Sandwich Theorem for functions immediately gives

$$\lim_{x \rightarrow 0} f(x) = 0 = f(0).$$

- (b) Using the usual rules, we find that, for $c \neq 0$,

$$f'(c) = 2c \sin \frac{1}{c} - \cos \frac{1}{c}.$$

- (c)

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = 0.$$

Therefore

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

(d) Let

$$g(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Since 0 is a limit point of the domain, to prove g is not continuous at 0 it suffices to prove

$$\lim_{x \rightarrow 0} g(x) \neq g(0).$$

In fact, $g(x)$ does not converge as x approaches 0. (For all $\delta > 0$, there is an $x_1 \in (0, \delta)$ such that $\cos \frac{1}{x_1} = 1$, and also $x_2 \in (0, \delta)$ such that $\cos \frac{1}{x_2} = -1$.)

Therefore $\lim_{x \rightarrow 0} g(x) \neq g(0)$, and g is not continuous at 0. □

9.8 (Smoothing out the corner). Consider the function $g : \mathbf{R} \rightarrow \mathbf{R}$ given by $g(x) = |x|$. We have seen in [Example 5.3](#) that g is not differentiable at 0.

(a) Fix $n \geq 1$. Find real numbers $a_n, b_n > 0$ such that the function $g_n : \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$g_n(x) = \begin{cases} x^{2n} + b_n & \text{if } |x| < a_n \\ |x| & \text{otherwise} \end{cases}$$

is continuous and differentiable on \mathbf{R} .

(b) Try to sketch the graphs of g_1, g_2, g_3 , and $|x|$ on the same graph (say between $x = -2$ and $x = 2$). What do you think the behaviour of the sequence (g_n) is as $n \rightarrow \infty$? Can you guess what the pointwise limit function is?

Solution.

(a) In order for g_n to be continuous and differentiable on \mathbf{R} , we must choose a_n and b_n so the following conditions are satisfied:

$$a_n^{2n} + b_n = a_n \quad \text{and} \quad 2na_n^{2n-1} = 1.$$

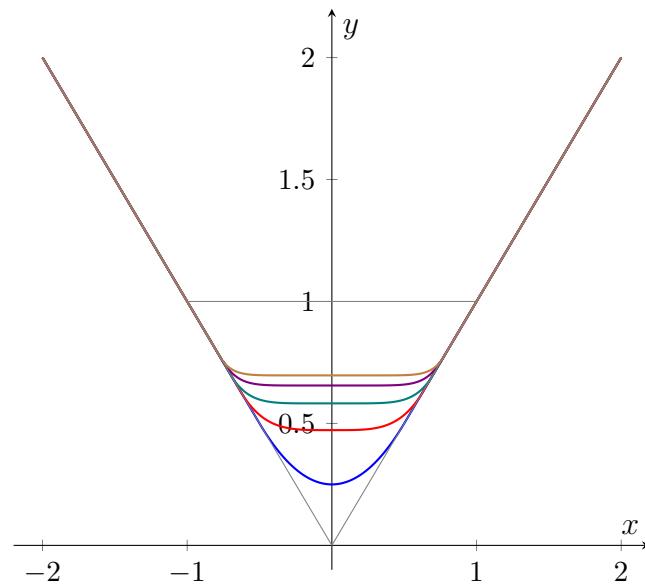
(The first ensures continuity by matching the values of the two functions that join to make up g_n , and the second ensures differentiability by matching the values of their derivatives.)

From these conditions we deduce that

$$a_n = \left(\frac{1}{2n}\right)^{1/(2n-1)} \quad \text{and} \quad b_n = a_n - a_n^{2n} = a_n \left(1 - \frac{1}{2n}\right).$$

(b) It is clear from the above that if a_n converges to a limit L , then b_n also converges to L . The sequence a_n does indeed converge, and $L = 1$. (We haven't quite seen enough to prove this, but you may be able to use Calculus 2 methods to see it.)

Here are the first few functions and the pointwise limit (assuming that indeed $a_n \rightarrow 1$):



The pointwise limit looks to be given by the function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ |x| & \text{otherwise.} \end{cases}$$

Can you prove it?

□