Topics: Mean Value Theorem

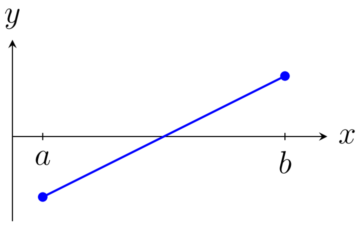
10.1 (Exploring the statement of Rolle's Theorem)**.** We explore why each of the hypotheses in Rolle's Theorem is required. To this end, in each of the following scenarios, draw a picture of a function that satisfies the given conditions, but has $f'(c) \neq 0$ for all $c \in (a, b)$.

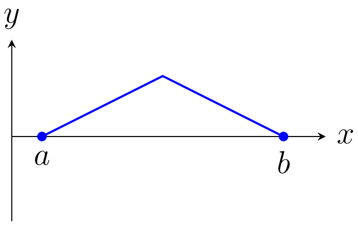
(a) f is continuous on $[a, b]$, differentiable on (a, b), but $f(a) \neq f(b)$.

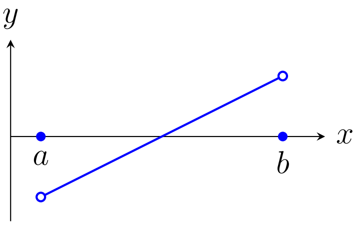
(b) f is continuous on $[a, b]$, has $f(a) = f(b) = 0$, but is not differentiable on (a, b).

(c) f is differentiable on (a, b), has $f(a) = f(b) = 0$, but f is not continuous on $[a, b]$.

Solution. Here are examples for each of the scenarios:







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10.2 (Exploring the statement of the Mean Value Theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b). For each of the criteria below, draw a picture of f satisfying it. If such f does not exist, briefly explain why not.

(a) For every $c \in (a, b)$ we have $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

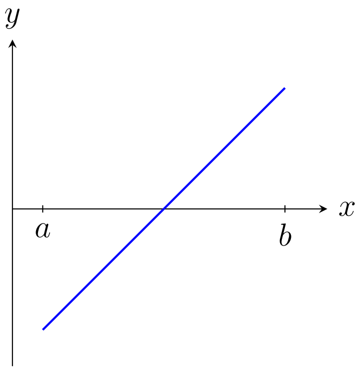
(b) There exists $c \in (a, b)$ so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

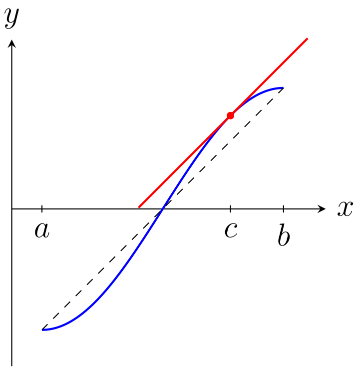
(c) For every $c \in (a, b)$ we have $f'(c) > \dfrac{f(b) - f(a)}{b - a}$.

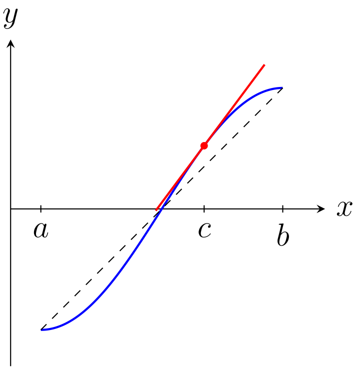
(d) There exists $c \in (a, b)$ so that $f'(c) > \dfrac{f(b) - f(a)}{b - a}$.

Solution. The type of function requested in part (c) does not exist, by the Mean Value Theorem.

For parts (a), (b), and (d), we have examples







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10.3 (Exploring the proof of the Mean Value Theorem)**.** Let f be continuous on $[a,b]$ and differentiable on (a,b). Recall the definition of s in the proof of the Mean Value Theorem:

$$s(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a).$$

Prove that if $s(x) = 0$ for all $x \in [a,b]$, then there exists $c \in (a,b)$ so that $f'(c) = 0$.

[**Hint**: Rolle's Theorem.]

Solution. Suppose $s(x) = 0$ for all $x \in [a,b]$. Then $0 = s(a) = f(a)$ and $0 = s(b) = f(b)$, and f is assumed to be continuous on $[a,b]$ and differentiable on (a,b). Therefore we may apply Rolle's Theorem to f and conclude that there exists $c \in (a,b)$ such that $f'(c) = 0$. ☐

10.4 (Use the Mean Value Theorem)**.** Let f be differentiable on $[a,b]$ and let $M \in \mathbf{R}$. Prove that if $f'(x) \geqslant M$ for all $x \in [a,b]$, then

$$M \leqslant \frac{f(b)-f(a)}{b-a}.$$

Solution. We prove the contrapositive: if $M > \dfrac{f(b)-f(a)}{b-a}$, then there exists $c \in [a,b]$ so that $f'(c) \leqslant M$.

By the Mean Value Theorem, there exists $c \in (a,b)$ so that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. And so if $\dfrac{f(b)-f(a)}{b-a} < M$, then $f'(c) < M$. ☐

10.5 (Contractive sequence)**.** Consider the sequence given by $x_0 = \frac{3}{2}$ and

$$x_{n+1} = -\frac{x_n^3}{12} + x_n + \frac{1}{4} \qquad \text{for all } n \in \mathbf{N}.$$

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by the formula

$$f(x) = -\frac{x^3}{12} + x + \frac{1}{4},$$

so that $x_{n+1} = f(x_n)$ for all $n \in \mathbf{N}$.

(a) Find the global minimum and global maximum of f on the interval $[1,2]$, and deduce that $1 \leqslant x_n \leqslant 2$ for all $n \in \mathbf{N}$.

(b) Use the Mean Value Theorem for f on the interval $[x_n, x_{n+1}]$, and deduce that (x_n) is a contractive sequence.

(c) Find the limit of (x_n).

Solution.

(a) Since f is continuous and differentiable on $[1,2]$, its global minimum and maximum occur either at the boundaries of the interval $[1,2]$, or at some stationary point in the interval. The only zero of $f'(x) = -\frac{x^2}{4} + 1$ in $[1,2]$ is $x = 2$, so we only need to evaluate f at 1 and 2:

$$f(1) = \frac{7}{6} \in [1,2], \qquad f(2) = \frac{19}{12} \in [1,2],$$

so $f\big([1,2]\big) \subseteq [1,2]$.

In particular, since $x_0 \in [1,2]$ and $x_{n+1} = f(x_n)$ for all $n \in \mathbf{N}$, we conclude that $x_n \in [1,2]$ for all $n \in \mathbf{N}$.

(b) We have

$$f'(x) = -\frac{x^2}{4} + 1,$$

and since $1 \leqslant x \leqslant 2$ it is easy to deduce that

$$0 \leqslant f'(x) \leqslant \frac{3}{4},$$

in particular $|f'(x)| \leqslant 3/4$ for all $x \in [1, 2]$.

Now let $n \in \mathbf{N}$ and consider $x_{n+1}, x_n \in [1, 2]$. Apply the Mean Value Theorem to f restricted to the interval $[x_n, x_{n+1}]$, and deduce that there exists $c \in (x_n, x_{n+1}) \subseteq [1, 2]$ such that

$$|x_{n+2} - x_{n+1}| = |f(x_{n+1}) - f(x_n)| = |f'(c)|\,|x_{n+1} - x_n| \leqslant \frac{3}{4}\,|x_{n+1} - x_n|,$$

in other words the sequence (x_n) is contractive with constant $3/4$.

(c) Since (x_n) is contractive, we know that it has a limit L. From part (b) we know that $1 \leqslant L \leqslant 2$. By the Algebra of Limits:

$$L = -\frac{L^3}{12} + L + \frac{1}{4} \Rightarrow L^3 = 3 \Rightarrow L = \sqrt[3]{3}. \qquad \square$$

Topics: Riemann integrals and Fundamental Theorem of Calculus

10.6 (Partitions and Riemann sums)**.** Consider $f:\ [-3,2] \longrightarrow \mathbf{R}$ with $f(x) = 4 - x^2$ and the partition $P = \left\{-3, -1, \frac{1}{2}, 2\right\}$.

(a) Find the lower sum $L(f,P)$.

(b) Find the upper sum $U(f,P)$.

(c) Using calculus knowledge, compute $\displaystyle\int_{-3}^{2} f(x)\,dx$ and compare it to the upper and lower sums. Does the result make sense?

Solution.

(a) The lower Riemann sum is

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{3} m_k(x_k - x_{k-1}) \\ &= -5 \cdot 2 + 3 \cdot \frac{3}{2} + 0 \cdot \frac{3}{2} \\ &= -\frac{11}{2}. \end{aligned}$$

(b) The upper Riemann sum is

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{3} M_k(x_k - x_{k-1}) \\ &= 3 \cdot 2 + 4 \cdot \frac{3}{2} + \frac{15}{4} \cdot \frac{3}{2} \\ &= \frac{141}{8}. \end{aligned}$$

(c) We have

$$\int_{-3}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-3}^{2} = \frac{25}{3}.$$

We see that

$$-\frac{11}{2} < \frac{25}{3} < \frac{141}{8}$$

as expected. ☐

10.7 (Integrating a constant function)**.** Let $k \in \mathbf{R}$ and let $f:\ [a,b] \longrightarrow \mathbf{R}$ with $f(x) = k$. Let $P = \left\{x_0, x_1, x_2, \ldots, x_n\right\}$ be a partition of $[a,b]$.

(a) Find expressions for the lower sum $L(f,P)$ and the upper sum $U(f,P)$.

(b) Find $L(f)$.

(c) Find $U(f)$.

(d) Determine if f is integrable. If it is integrable, find $\displaystyle\int_{a}^{b} f(x)\,dx$.

Solution.

(a) As $f(x)$ is constant then for each interval $[x_{i-1}, x_i]$ we have $m_i = M_i = k$. The lower sum for this partition is

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} k(x_i - x_{i-1}) \\
&= (kx_1 - kx_0) + (kx_2 - kx_1) + (kx_3 - kx_2) + \cdots + (kx_n - kx_{n-1}) \\
&= -kx_0 + (kx_1 - kx_1) + (kx_2 - kx_2) + (kx_3 - kx_3) + \cdots + (kx_{n-1} - kx_{n-1}) + kx_n \\
&= kb - ka.
\end{aligned}$$

The upper sum is

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} k(x_i - x_{i-1}) \\
&= (kx_1 - kx_0) + (kx_2 - kx_1) + (kx_3 - kx_2) + \cdots + (kx_n - kx_{n-1}) \\
&= -kx_0 + (kx_1 - kx_1) + (kx_2 - kx_2) + (kx_3 - kx_3) + \cdots + (kx_{n-1} - kx_{n-1}) + kx_n \\
&= kb - ka.
\end{aligned}$$

(b) As we had an arbitrary partition then all partitions have a lower sum of $k(b-a)$, so $L(f) = k(b-a)$.

(c) Reasoning as above, $U(f) = k(b-a)$.

(d) The function $f(x) = k$ is integrable and $U(f) = L(f) = \int_a^b f(x)\,dx = k(b-a)$. ☐

10.8 (Bounding Riemann integrals)**.** Let $f : [a,b] \longrightarrow \mathbf{R}$ be a function and let

$$A_f = \{f(x) : x \in [a,b]\}.$$

be the image of f. Suppose that f is bounded, in other words that the set A_f is bounded.

(a) Prove that
$$(b-a) \inf A_f \leqslant L(f).$$

(b) What do you think the correct analogous statement for $U(f)$ is?

Solution.

(a) Consider the partition $Q = \{x_0 = a, x_1 = b\}$ of $[a,b]$. Then $m_1 = \inf A_f$, so

$$L(f,Q) = m_1(x_1 - x_0) = \big(\inf A_f\big)(b-a).$$

As $L(f)$ is the supremum of $L(f,P)$ over all partitions P, we have in particular $L(f) \geqslant L(f,Q) = (b-a)\inf A_f$.

(b) $U(f) \leqslant (b-a) \sup A_f$. ☐

10.9 (One point does not matter I)**.** Let $c \in [a,b]$ and consider the function $h : \ [a,b] \longrightarrow \mathbf{R}$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x \neq c, \\ 1 & \text{if } x = c. \end{cases}$$

Prove that h is integrable and $\displaystyle\int_a^b h(x)\,dx = 0$.

[**Hint**: Use Theorem 5.29 to prove integrability, then determine $L(f,P)$ for an arbitrary partition P of $[a,b]$.]

Solution. Let $\varepsilon > 0$. We have three cases:

- $a < c < b$. Then let $P_\varepsilon = \{x_0 = a, x_1 = c - \varepsilon/3, x_2 = c + \varepsilon/3, x_3 = b\}$. We have

m_1	m_2	m_3	M_1	M_2	M_3
0	0	0	0	1	0

  Putting it together:

  $$U(f,P_\varepsilon) - L(f,P_\varepsilon) = M_2(x_2 - x_1) - 0 = \frac{2\varepsilon}{3} < \varepsilon.$$

- $a = c$. Then let $P_\varepsilon = \{x_0 = a = c, x_1 = c + \varepsilon/2, x_2 = b\}$. We have

m_1	m_2	M_1	M_2
0	0	1	0

  Putting it together:

  $$U(f,P_\varepsilon) - L(f,P_\varepsilon) = M_1(x_1 - x_0) - 0 = \frac{\varepsilon}{2} < \varepsilon.$$

- $b = c$, similar to the previous case.

By Theorem 5.29 we conclude that h is integrable on $[a,b]$.

But $L(h,P) = 0$ for every partition P of $[a,b]$, so $L(h) = 0$, hence $\displaystyle\int_a^b h(x)\,dx = 0$. ☐

10.10 (One point does not matter II)**.** Let $f : \ [a,b] \longrightarrow \mathbf{R}$ be an integrable function and let $c \in [a,b]$. Let $r \in \mathbf{R}$ and consider the function $g : \ [a,b] \longrightarrow \mathbf{R}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq c, \\ r & \text{if } x = c. \end{cases}$$

(a) Use Tutorial Question 10.9 and Theorem 5.31 to show that g is integrable on $[a,b]$ and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

(b) Consider the slogan "Finitely many points do not matter". Turn it into a precise mathematical statement generalising part (b), then prove the statement.

Solution.

(a) In terms of the function h defined in Tutorial Question 10.9, we have

$$g(x) = f(x) + \big(r - f(c)\big)h(x) \qquad \text{for all } x \in [a, b].$$

Now apply Theorem 5.31 to the above and get that g is integrable and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx + \big(r - f(c)\big)\int_a^b h(x)\,dx = \int_a^b f(x)\,dx.$$

(b) Here is one way to state the generalisation: "Let $k \in \mathbf{N}$ with $k \geqslant 1$ and pick finitely many points $c_1, \ldots, c_k \in [a, b]$. Let $g\colon\ [a, b] \longrightarrow \mathbf{R}$ be a function that agrees with f at all $x \in [a, b] \setminus \{c_1, \ldots, c_k\}$. Then g is integrable and $\displaystyle\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$."

The proof is by induction.

Base case $k = 1$: this is precisely part (a).

Induction step: let $k \geqslant 1$ be arbitrary but fixed and suppose that the statement holds for this value of k.

Let $c_1, \ldots, c_k, c_{k+1} \in [a, b]$ and let $g\colon\ [a, b] \longrightarrow \mathbf{R}$ be a function that agrees with f at all $x \in [a, b] \setminus \{c_1, \ldots, c_k, c_{k+1}\}$.

Define the function $\widehat{g}\colon\ [a, b] \longrightarrow \mathbf{R}$ by

$$\widehat{g}(x) = \begin{cases} g(x) & \text{if } x \neq c_{k+1}, \\ f(c_{k+1}) & \text{if } x = c_{k+1}. \end{cases}$$

Then $\widehat{g}$ agrees with f at all $x \in [a, b] \setminus \{c_1, \ldots, c_k\}$, hence by the induction hypothesis it is integrable and $\displaystyle\int_a^b \widehat{g}(x)\,dx = \int_a^b f(x)\,dx$.

On the other hand, g agrees with $\widehat{g}$ at all $x \in [a, b] \setminus \{c_{k+1}\}$, so by the base case we have that g is integrable and $\displaystyle\int_a^b g(x)\,dx = \int_a^b \widehat{g}(x)\,dx$. $\square$