Question 1. Let (X, d) be a metric space.

(a) Define the concept "D is a dense subset of X".

(b) Show that $D \subseteq X$ is a dense subset of X if and only if $D \cap U \neq \varnothing$ for all nonempty open sets U in X.

(c) Prove that the intersection of two dense open sets U_1 and U_2 is dense.

Solution:

(a) $X = \overline{D}$.

(b) Suppose D is dense and let U be nonempty open. Let $x \in U$. As U is open, there exists $\mathbf{B}_r(x) \subseteq U$ with $r > 0$. If $x \in D$, we are done. Otherwise, $x \in X \setminus D = \overline{D} \setminus D$, so it is a limit point of D, so there exists $a \in \mathbf{B}_r(x) \cap D$ such that $a \neq x$, hence $a \in U \cap D$.

Conversely, suppose $D \cap U$ is nonempty for any nonempty open U. Let $x \in X \setminus D$. For every $r > 0$, $U := \mathbf{B}_r(x)$ is open so $D \cap \mathbf{B}_r(x)$ is nonempty, and x is not in this intersection so there must be a point distinct from x in it, hence $x \in \overline{D}$.

(c) Let $U_{12} = U_1 \cap U_2$.

To show that U_{12} is dense, we use the previous part and show that $U_{12} \cap U \neq \varnothing$ for all nonempty open U:

$$U_{12} \cap U = \big(U_1 \cap U_2\big) \cap U = U_1 \cap \big(U_2 \cap U\big).$$

Since U_2 is dense and open, $U_2 \cap U$ is nonempty and open. Since U_1 is dense, $U_1 \cap \big(U_2 \cap U\big)$ is nonempty. So $U_{12} \cap U \neq \varnothing$, hence U_{12} is dense.

Question 2. Let (X, d) be a metric space.

(a) Define the concept "D is a disconnected subset of X".

(b) Prove that a subset D of X is disconnected if and only if there exists a surjective continuous function $g\colon D \longrightarrow \{0,1\}$, where $\{0,1\}$ is given the discrete metric.

(c) Suppose $A \subseteq X$ is a connected subset and $\{C_i\colon i \in I\}$ is an arbitrary collection of connected subsets of X such that $A \cap C_i \neq \varnothing$ for all $i \in I$. Prove that

$$B := A \cup \bigcup_{i \in I} C_i$$

is a connected subset of X.

Solution:

(a) There exist open subsets U and V of D such that

$$D = U \cup V, \qquad U \cap V = \varnothing, \qquad U \neq \varnothing, \qquad V \neq \varnothing.$$

(b) If such a function g exists, let $U = g^{-1}(0)$ and $V = g^{-1}(1)$, then $U \neq \varnothing$, $V \neq \varnothing$ since g is surjective. As $\{0\} \cap \{1\} = \varnothing$, we have $U \cap V = \varnothing$. Clearly $D = U \cup V$, and both U and V are open since $\{0\}$ and $\{1\}$ are open. This implies that D is disconnected.

For the other direction, suppose that D is disconnected and write $D = U \cup V$ with U, V as in the definition. Define $g\colon X \longrightarrow \{0,1\}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V. \end{cases}$$

This is well-defined since $U \cap V = \varnothing$. It is continuous as $g^{-1}(0) = U$ and $g^{-1}(1) = V$ are open. It is surjective since it takes both values 0 and 1 (as both U and V are nonempty).

(c) Let $g\colon B \longrightarrow \{0,1\}$ be an arbitrary continuous function.

Its restriction $g|_A\colon A \longrightarrow \{0,1\}$ cannot be surjective, since A is connected. So $g|_A$ is constant, let's say 0 for concreteness.

Now let $i \in I$. The restriction $g|_{C_i}\colon C_i \longrightarrow \{0,1\}$ must be constant, for the same reason as before. But $A \cap C_i \neq \varnothing$ and g is zero on A, so g must be zero on C_i.

As this holds for all $i \in I$, we conclude that g is zero on B.

So there is no surjective continuous map $B \longrightarrow \{0,1\}$, hence B must be connected.

Question 3. Let (X, d) be a metric space.

(a) Define the concept "K is a compact subset of X".

(b) Let C be a closed subset of a compact subset K of X. Prove that C is compact.

(c) Let K and L be compact subsets of X. Prove that $K \cup L$ is compact.

Solution:

(a) Given any open cover of K:

$$K \subseteq \bigcup_{i \in I} U_i,$$

there exists a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$K \subseteq \bigcup_{j=1}^{n} U_{i_j}.$$

(b) Consider an arbitrary open cover of C:

$$C \subseteq \bigcup_{i \in I} U_i.$$

Then we have

$$K \subseteq X = C \cup (X \setminus C) \subseteq \left(\bigcup_{i \in I} U_i \right) \cup (X \setminus C),$$

which is an open cover of K. As K is compact, there is a finite subcover, so that

$$K \subseteq \left(\bigcup_{n=1}^{N} U_{i_n} \right) \cup (X \setminus C), \qquad i_n \in I,$$

hence

$$C \subseteq \bigcup_{n=1}^{N} U_{i_n}.$$

(c) Consider an arbitrary open cover of $K \cup L$:

$$K \cup L \subseteq \bigcup_{i \in I} U_i.$$

This is also an open cover of K, so there is a finite subcover that still covers K:

$$K \subseteq \bigcup_{n=1}^{N} U_{i_n}, \qquad i_n \in I.$$

Similarly, we get a finite subcover that covers L:

$$L \subseteq \bigcup_{m=1}^{M} U_{j_m}, \qquad j_m \in I.$$

Letting $S = \{i_1, \ldots, i_N\} \cup \{j_1, \ldots, j_M\}$, we get a finite subcover that covers $K \cup L$:

$$K \cup L \subseteq \bigcup_{s \in S} U_s.$$

Question 4. Consider the equation

$$x^3 - x - 1 = 0. \tag{1}$$

(a) Show that Equation (1) must have **at least one solution** in the interval $[1, 2]$.

(b) Show that the function $f: [1, 2] \longrightarrow [1, 2]$ given by

$$f(x) = (1 + x)^{1/3}$$

is a contraction.

(c) Show that Equation (1) has **a unique solution** ξ in the interval $[1, 2]$ and describe a sequence of real numbers that converges to ξ.

Solution:

(a) We can use the Intermediate Value Theorem, as $x^3 - x - 1$ is clearly continuous. At $x = 1$, $x^3 - x - 1 = -1 < 0$, while at $x = 2$, $x^3 - x - 1 = 5 > 0$, so there must be at least one point x in $[1, 2]$ such that $x^3 - x - 1 = 0$.

(b) The derivative of f is

$$f'(x) = \frac{1}{3}(1 + x)^{-2/3} = \frac{1}{3}\frac{1}{(1 + x)^{2/3}}.$$

As $x \in [1, 2]$, we have $f'(x) > 0$ and

$$1 \leqslant x \Rightarrow 2 \leqslant 1 + x \Rightarrow \frac{1}{1 + x} \leqslant \frac{1}{2} \Rightarrow \frac{1}{(1 + x)^{2/3}} \leqslant \frac{1}{2^{2/3}} \leqslant 1,$$

so that

$$f'(x) \leqslant \frac{1}{3}.$$

Now let x, y be such that $1 \leqslant x < y \leqslant 2$ and apply the Mean Value Theorem to f on $[x, y]$ to deduce that there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \Rightarrow |f(y) - f(x)| = |f'(c)|\,|y - x| \leqslant \frac{1}{3}|y - x|.$$

We conclude that f is a contraction.

(c) Observe that $x^3 - x - 1 = 0$ is equivalent to $f(x) = x$, so the solutions of Equation (1) are precisely the fixed points of f. As f is a contraction and $[1, 2]$ is complete, the Banach Fixed Point Theorem says that there is a unique fixed point ξ in $[1, 2]$. It also tells us that we can start with any $x_1 \in [1, 2]$, for instance $x_1 = 1$, and iteratively apply f to get a sequence (x_n) converging to ξ:

$$x_1 = 1, \qquad x_2 = f(x_1) = 2^{1/3}, \qquad x_3 = f(x_2) = \left(1 + 2^{1/3}\right)^{1/3}, \ldots$$

Question 5. (a) Let $f \in L(V, W)$ be a continuous linear map between normed spaces. Prove that if U is a closed subspace of W, then its preimage $f^{-1}(U)$ is a closed subspace of V.

(b) Prove that the following set of sequences

$$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}$$

is a closed subspace of the Banach space ℓ^1:

$$\ell^1 = \left\{ (a_n) \in \mathbf{F}^{\mathbf{N}} : \sum_{n=1}^{\infty} |a_n| < \infty \right\}.$$

Solution:

(a) Clear since f is linear so the inverse image of a subspace is a subspace; and f is continuous so the inverse image of a closed set is a closed set.

(b) Consider the function $f \colon \ell^1 \longrightarrow \mathbf{F}$ given by

$$f\big((a_n)\big) = \sum_{n=1}^{\infty} a_n.$$

First note that this is a reasonable definition, because the infinite series on the right hand side converges in $\mathbf{F}$:

$$\left| \sum_{n=1}^{N} a_n \right| \leqslant \sum_{n=1}^{N} |a_n|,$$

and the latter converges as $N \longrightarrow \infty$ since $(a_n) \in \ell^1$.

The function f is linear. It is also continuous, because as we have just seen:

$$\big| f\big((a_n)\big) \big| = \left| \sum_{n=1}^{\infty} a_n \right| \leqslant \sum_{n=1}^{\infty} |a_n| = \|(a_n)\|_{\ell^1}.$$

Hence $f \in L(\ell^1, \mathbf{F}) = (\ell^1)^\vee$ and its kernel is S, so S is a closed subspace of ℓ^1.

Question 6. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

(a) Given a subset S of V, define the concept "the orthogonal complement $S^\perp$ of S".

(b) Prove that $S \subseteq \left(S^\perp\right)^\perp$.

(c) Prove that if V is a Hilbert space and W is a closed subspace of V, then $\left(W^\perp\right)^\perp = W$.

Solution:

(a) $S^\perp = \{v \in V : \langle v, s\rangle = 0 \text{ for all } s \in S\}$.

(b) Let $s \in S$. For any $x \in S^\perp$, we have

$$\langle s, x\rangle = \overline{\langle x, s\rangle} = 0,$$

so $s \in \left(S^\perp\right)^\perp$.

(c) We have seen above that $W \subseteq \left(W^\perp\right)^\perp$.

Let $x \in \left(W^\perp\right)^\perp$. By the Hilbert Projection Theorem, we can decompose

$$H = W \oplus W^\perp.$$

So we have $x = y + z$ with $y \in W$ and $z \in W^\perp$. Then

$$0 = \langle x, z\rangle = \langle y + z, z\rangle = \langle y, z\rangle + \langle z, z\rangle = 0 + \|z\|^2,$$

implying that $z = 0$ and $x = y \in W$.

Question 7. (a) State the Cauchy–Schwarz Inequality for inner product spaces.

(b) Let V be an inner product space. Prove that for any $u \in V$ we have

$$\|u\| = \sup_{\|v\|=1} \left|\langle u, v\rangle\right|.$$

(c) Now let W be a second inner product space and let $f \in L(V, W)$ be a continuous linear map. Prove that

$$\|f\| = \sup_{\|v\|_V = 1 = \|w\|_W} \left|\langle f(v), w\rangle_W\right|.$$

Solution:

(a) For any u, v in an inner product space V we have

$$\left|\langle u, v\rangle\right| \leqslant \|u\| \, \|v\|.$$

Equality holds if and only if u and v are linearly dependent.

(b) If $u = 0$ then the equality is obvious.

So assume now that $u \neq 0$. Applying Cauchy–Schwarz with $v \in V$ such that $\|v\| = 1$, we have

$$\left|\langle u, v\rangle\right| \leqslant \|u\|,$$

so that

$$\sup_{\|v\|=1} \left|\langle u, v\rangle\right| \leqslant \|u\|.$$

To get equality, take $v = \frac{1}{\|u\|} u$ and see that the LHS is indeed $\|u\|$.

(c) From the previous part:

$$\|u\|_W = \sup_{\|w\|_W = 1} \left|\langle u, w\rangle_W\right| \qquad \text{for all } u \in W.$$

Setting $u = f(v)$ for some $v \in V$, we get

$$\|f(v)\|_W = \sup_{\|w\|_W = 1} \left|\langle f(v), w\rangle_W\right| \qquad \text{for all } v \in V.$$

Therefore

$$\|f\| = \sup_{\|v\|_V = 1} \|f(v)\|_W = \sup_{\|v\|_V = 1 = \|w\|_W} \left|\langle f(v), w\rangle_W\right|.$$

Question 8. Consider the function $g\colon \ell^2 \longrightarrow \mathbf{F}$ given by

$$g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n^2}.$$

(a) Find $y \in \ell^2$ such that

$$g(x) = \langle x, y \rangle \qquad \text{for all } x \in \ell^2.$$

(b) Deduce that g is linear and continuous and find its norm $\|g\|$.

[*Hint*: You may use without proof the fact that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.]

Solution:

(a) Setting $y = (y_n)$ with

$$y_n = \frac{1}{n^2},$$

we certainly have for all $x = (x_n) \in \ell^2$:

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n = \sum_{n=1}^{\infty} \frac{x_n}{n^2} = g(x).$$

We should check that $y \in \ell^2$:

$$\|y\|_{\ell^2}^2 = \sum_{n=1}^{\infty} y_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(b) From the previous part we know that $g = y^\vee$, so certainly g is linear and continuous. We also have

$$\|g\| = \|y^\vee\| = \|y\|_{\ell^2} = \frac{\pi^2}{3\sqrt{10}},$$

as we have seen in the previous part.

Question 9. Let (a_n) be a decreasing sequence of non-negative real numbers. Consider $f\colon \ell^2 \longrightarrow \mathbf{F}^{\mathbf{N}}$ given by

$$f(x) = \big(a_1x_1, a_2x_2, \dots, a_nx_n, \dots\big).$$

(a) Prove that the image of f is contained in ℓ^2 and that $f\colon \ell^2 \longrightarrow \ell^2$ is linear and continuous.

(b) Find the norm $\|f\|$.

(c) Find the adjoint f^* of f.

(d) How much can you relax the conditions on the sequence (a_n) and still retain the statement in part (a)? Make an educated guess and describe briefly how/if the answers to parts (b) and (c) change.

Solution:

(a) We have

$$\|f(x)\|_{\ell^2}^2 = \sum_{n=1}^{\infty} a_n^2\,|x_n|^2 \leqslant a_1^2 \sum_{n=1}^{\infty} |x_n|^2 = a_1^2\,\|x\|_{\ell^2}^2,$$

so if $x \in \ell^2$ then $f(x) \in \ell^2$.

It is straightforward that f is linear. It is clear that f is continuous from the inequality above.

(b) We have

$$\|f\| = \sup_{\|x\|=1} \big\|f(x)\big\| \leqslant a_1$$

from the previous part.

Taking $x = e_1 = (1,0,0,\dots)$ we have $\|e_1\| = 1$ and $f(e_1) = (a_1,0,0,\dots)$ so $\|f(e_1)\| = a_1$, therefore $\|f\| = a_1$.

(c) We have

$$\big\langle f(x), y\big\rangle = \sum_{n=1}^{\infty} a_n\,x_n\,\overline{y}_n = \sum_{n=1}^{\infty} x_n\,\overline{(a_n\,y_n)} = \big\langle x, f(y)\big\rangle,$$

where we used the fact that $a_n \in \mathbf{R}$ for all $n \in \mathbf{N}$.

Therefore $f^* = f$.

(d) We can take (a_n) to be any bounded sequence of complex numbers and (a) still holds. In (b) we get $\|f\| = \|(a_n)\|_{\ell^\infty}$, and in (c) we get

$$f^*(y) = \big(\overline{a}_1y_1, \overline{a}_2y_2, \dots, \overline{a}_ny_n, \dots\big).$$