Question 1. For each $n \in \mathbf{N}$ define $f_n \colon [0,1] \longrightarrow \mathbf{R}$ by

$$f_n(x) = \frac{1 - x^n}{1 + x^n}.$$

(a) Show that f_n is continuous and bounded for all $n \in \mathbf{N}$.

(b) Define the concept: the sequence $\left(f_n\right)$ converges pointwise to $f \colon [0,1] \longrightarrow \mathbf{R}$.

Find the pointwise limit f of the sequence $\left(f_n\right)$.

(c) Define the concept: the sequence $\left(f_n\right)$ converges uniformly to $f \colon [0,1] \longrightarrow \mathbf{R}$.

Determine whether the sequence $\left(f_n\right)$ converges uniformly to the pointwise limit f from part (b).

Question 2. (a) Define the notion of adjoint map of a continuous linear map $f\colon X \longrightarrow Y$ between Hilbert spaces.

(b) Let $f\colon X \longrightarrow Y$ be a continuous linear map between Hilbert spaces. Prove that

$$\ker\left(f^*\right) = \left(\operatorname{im} f\right)^{\perp}.$$

(c) Give an example of a Hilbert space H and a continuous linear map $f\colon H \longrightarrow H$ such that

$$H = \operatorname{im} f \oplus \ker f$$

and neither $\operatorname{im} f$ nor $\ker f$ is the zero space.

Question 3. Let $f\colon X \longrightarrow Y$ be a continuous function between metric spaces.

(a) Define the concept: f is uniformly continuous.

(b) Prove that if X is compact, then f is uniformly continuous.

(c) Prove that if $f\colon X \longrightarrow Y$ is uniformly continuous and (x_n) is a Cauchy sequence in X, then $\big(f(x_n)\big)$ is a Cauchy sequence in Y.

(d) Give an example of a continuous function $f\colon X \longrightarrow Y$ that is not uniformly continuous.

Question 4. Consider the Hilbert space ℓ^2 of square-summable sequences $a = (a_1, a_2, \dots)$ and let $\{e_1, e_2, \dots\}$ with $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, …, be the standard Schauder basis for ℓ^2.

Let $f\colon \ell^2 \longrightarrow \ell^2$ be a linear transformation.

(a) Show that if f is a continuous linear map then the sequence $\big(\|f(e_n)\|\big)$ is bounded.

(b) Writing

$$f(e_n) = \sum_{m=1}^{\infty} c_{nm}\, e_m,$$

give a condition on the coefficients c_{nm} that is necessary and sufficient for f to be self-adjoint.

Question 5. Let X be a topological space and let K, L be two subsets of X.

(a) Define the concepts: (i) X is Hausdorff; (ii) K is compact.

(b) Prove that if X is a Hausdorff topological space and K is a compact subset of X, then K is closed in X.

(c) Prove that if X is a compact topological space and $K \subseteq X$ is a closed subset, then K is compact.

(d) Suppose K and L are compact subsets of a Hausdorff topological space X. Prove that the intersection $K \cap L$ is compact.

Question 6. Recall that $C_0([0,1],\mathbf{R})$ denotes the space of (bounded) continuous functions $f\colon [0,1] \longrightarrow \mathbf{R}$.

(a) State the Weierstrass Approximation Theorem for $C_0([0,1],\mathbf{R})$ with the uniform norm.

(b) Suppose $f \in C_0([0,1],\mathbf{R})$ has the property that

$$\int_0^1 f(x)\, x^n\, dx = 0 \qquad \text{for all } n = 0, 1, 2, \ldots. \tag{1}$$

Prove that f is the constant function 0 on $[0,1]$.

(c) Give an explicit **discontinuous** function $f\colon [0,1] \longrightarrow \mathbf{R}$ that satisfies equation (1) but is (obviously) not the constant function 0 on $[0,1]$.

Question 7. In this question, we endow the product $S \times T$ of any two metric spaces S and T with its Manhattan metric:

$$d\big((s_1, t_1), (s_2, t_2)\big) = d_S(s_1, s_2) + d_T(t_1, t_2).$$

(a) Prove that if S and T are complete metric spaces, then the metric space $S \times T$ is complete.

(b) Define the concept of completion of a metric space X.

(c) Let X, Y be metric spaces and fix completions $\big(\widehat{X}, \iota_X\big)$ of X and $\big(\widehat{Y}, \iota_Y\big)$ of Y.

Prove that $\big(\widehat{X} \times \widehat{Y}, \iota_X \times \iota_Y\big)$ is a completion of $X \times Y$.

Question 8. (a) Let X be a metric space.

Define the concepts: (i) X is **complete**; (ii) $f\colon X \longrightarrow X$ is a **contraction**; (iii) $x \in X$ is a **fixed point** of $f\colon X \longrightarrow X$.

Give the complete statement of the Banach Fixed Point Theorem.

(b) Consider the function $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$. Find a positive real number $a > 0$ such that f satisfies the **hypotheses** of the Banach Fixed Point Theorem on the interval $[-a, a]$.

Give your best guess for how large you can make a (without proof).

(c) What is the largest interval on which the **conclusion** of the Banach Fixed Point Theorem holds for the same function $f(x) = x^2$?