Question 1. For each $n \in \mathbf{N}$ define $f_n \colon [0,1] \longrightarrow \mathbf{R}$ by

$$f_n(x) = \frac{1 - x^n}{1 + x^n}.$$

(a) Show that f_n is continuous and bounded for all $n \in \mathbf{N}$.

(b) Define the concept: the sequence (f_n) converges pointwise to $f \colon [0,1] \longrightarrow \mathbf{R}$.

Find the pointwise limit f of the sequence (f_n).

(c) Define the concept: the sequence (f_n) converges uniformly to $f \colon [0,1] \longrightarrow \mathbf{R}$.

Determine whether the sequence (f_n) converges uniformly to the pointwise limit f from part (b).

Solution:

(a) Both $1 - x^n$ and $1 + x^n$ are continuous, and $1 + x^n$ is nonzero on $[0,1]$, so the quotient f_n is continuous on $[0,1]$.

It is then clear that f_n is bounded, as it is continuous on the compact domain $[0,1]$; or we can bound it explicitly: for $0 \leqslant x \leqslant 1$ we have $0 \leqslant 1 - x^n \leqslant 1$ and $1 \leqslant 1 + x^n \leqslant 2$, so

$$0 \leqslant \frac{1 - x^n}{1 + x^n} \leqslant 1.$$

(b) The sequence (f_n) converges pointwise to $f \colon [0,1] \longrightarrow \mathbf{R}$ if for every $x \in [0,1]$ the sequence of real numbers $(f_n(x))$ converges to the real number $f(x)$.

Alternative: for every $x \in [0,1]$ and every $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $n \geqslant N$ we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Note that at $x = 1$ we have $f_n(1) = \frac{0}{2} = 0$, so $f(1) = 0$.

But if $x < 1$ then $(x^n) \longrightarrow 0$ as $n \longrightarrow \infty$, so that

$$f_n(x) = \frac{1 - x^n}{1 + x^n} \longrightarrow \frac{1}{1} = 1,$$

and so $f(x) = 1$. In summary:

$$f(x) = \begin{cases} 1 & \text{if } 0 \leqslant x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

(c) The sequence (f_n) converges uniformly to $f \colon [0,1] \longrightarrow \mathbf{R}$ if the sequence of real numbers $(\|f_n - f\|_\infty)$ converges to 0, where for any continuous function $g \colon [0,1] \longrightarrow \mathbf{R}$ we have

$$\|g\|_\infty = \sup_{x \in [0,1]} |g(x)|.$$

Equivalently: for every $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $n \geqslant N$ we have

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } x \in [0,1].$$

As the f_n are all continuous, if $(f_n) \longrightarrow f$ uniformly then f would be continuous. Since that is not the case, the convergence is not uniform.

Question 2. (a) Define the notion of adjoint map of a continuous linear map $f\colon X \longrightarrow Y$ between Hilbert spaces.

(b) Let $f\colon X \longrightarrow Y$ be a continuous linear map between Hilbert spaces. Prove that

$$\ker\left(f^*\right) = \left(\operatorname{im} f\right)^{\perp}.$$

(c) Give an example of a Hilbert space H and a continuous linear map $f\colon H \longrightarrow H$ such that

$$H = \operatorname{im} f \oplus \ker f$$

and neither $\operatorname{im} f$ nor $\ker f$ is the zero space.

Solution:

(a) The adjoint $f^*\colon Y \longrightarrow X$ is the unique element of $L(Y, X)$ with the property that

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle \qquad \text{for all } x \in X, y \in Y.$$

Alternative: let $\Phi_X\colon X \longrightarrow X^\vee$ and $\Phi_Y\colon Y \longrightarrow Y^\vee$ be the conjugate-linear isometries given by the Riesz Representation Theorem and let $f^\vee\colon Y^\vee \longrightarrow X^\vee$ be the dual linear transformation of f, then the adjoint $f^*\colon Y \longrightarrow X$ is defined by

$$f^* = \Phi_X^{-1} \circ f^\vee \circ \Phi_Y^{-1}.$$

(b) We have

$$\begin{aligned} y \in \left(\operatorname{im} f\right)^{\perp} &\iff y \perp f(x) \qquad \text{for all } x \in X \\ &\iff \left\langle f(x), y\right\rangle = 0 \qquad \text{for all } x \in X \\ &\iff \left\langle x, f^*(y)\right\rangle = 0 \qquad \text{for all } x \in X \\ &\iff f^*(y) = 0 \\ &\iff y \in \ker f^*. \end{aligned}$$

(c) Let $H = \mathbf{C}^2$ and $f\colon \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ given by multiplication by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The claim follows from the Hilbert Projection Theorem since $H = \operatorname{im} f \oplus \left(\operatorname{im} f\right)^{\perp} = \operatorname{im} f \oplus \ker f^* = \operatorname{im} f \oplus \ker f$ as $f = f^*$ is self-adjoint.

Or we can check directly that $\operatorname{im} f = \mathbf{C}e_1$ and $\ker f = \mathbf{C}e_2$, hence the claim.

Question 3. Let $f\colon X \longrightarrow Y$ be a continuous function between metric spaces.

(a) Define the concept: f is uniformly continuous.

(b) Prove that if X is compact, then f is uniformly continuous.

(c) Prove that if $f\colon X \longrightarrow Y$ is uniformly continuous and (x_n) is a Cauchy sequence in X, then $\big(f(x_n)\big)$ is a Cauchy sequence in Y.

(d) Give an example of a continuous function $f\colon X \longrightarrow Y$ that is not uniformly continuous.

Solution:

(a) We say that $f\colon X \longrightarrow Y$ is uniformly continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \quad d_X(x, x') < \delta \quad \text{then} \quad d_Y\big(f(x), f(x')\big) < \varepsilon.$$

(b) Let $\varepsilon > 0$.

Given $x \in X$, there exists $\delta(x) > 0$ such that $f\big(\mathbf{B}_{\delta(x)}(x)\big) \subseteq \mathbf{B}_{\varepsilon/2}\big(f(x)\big)$. We get an open cover of X:

$$X \subseteq \bigcup_{x \in X} \mathbf{B}_{\delta(x)/2}(x),$$

which therefore has a finite subcover

$$X \subseteq \bigcup_{n=1}^{N} \mathbf{B}_{\delta(x_n)/2}(x_n).$$

Let $\delta = \min\big\{\delta(x_n)/2 \colon n = 1, \dots, N\big\}$.

Suppose $s, t \in X$ are such that $d_X(s,t) < \delta$. We have $s \in \mathbf{B}_{\delta(x_n)/2}(x_n)$ for some $n \in \{1, \dots, N\}$. I claim that $t \in \mathbf{B}_{\delta(x_n)}(x_n)$:

$$d_X(t, x_n) \leqslant d_X(t, s) + d_X(s, x_n) < \delta + \frac{\delta(x_n)}{2} \leqslant \delta(x_n).$$

Therefore $f(s), f(t) \in \mathbf{B}_{\varepsilon/2}\big(f(x_n)\big)$, hence $d_Y(f(s), f(t)) < \varepsilon$.

(c) For all $n \in \mathbf{N}$, set $y_n = f(x_n)$.

Let $\varepsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that for all $x, x' \in X$, if $d_X(x, x') < \delta$ then $d_Y\big(f(x), f(x')\big) < \varepsilon$.

But (x_n) is Cauchy in X, so given this δ there exists $N \in \mathbf{N}$ such that $d_X(x_n, x_m) < \delta$ for all $n, m \geqslant N$. Therefore $d_Y\big(y_n, y_m\big) < \varepsilon$ for all $n, m \geqslant N$.

(d) Take $f\colon (0,1) \longrightarrow \mathbf{R}$ given by $f(x) = \frac{1}{x}$, then f is continuous on $(0,1)$, but it maps the Cauchy sequence $(1/n)$ to the sequence (n), which is not Cauchy.

Question 4. Consider the Hilbert space ℓ^2 of square-summable sequences $a = (a_1, a_2, \dots)$ and let $\{e_1, e_2, \dots\}$ with $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, ..., be the standard Schauder basis for ℓ^2.

Let $f \colon \ell^2 \longrightarrow \ell^2$ be a linear transformation.

(a) Show that if f is a continuous linear map then the sequence $\big(\|f(e_n)\|\big)$ is bounded.

(b) Writing

$$f(e_n) = \sum_{m=1}^{\infty} c_{nm}\, e_m,$$

give a condition on the coefficients c_{nm} that is necessary and sufficient for f to be self-adjoint.

Solution:

(a) We know that a linear map f is continuous if and only if it is Lipschitz, that is there exists $C > 0$ such that

$$\|f(v)\| \leqslant C\|v\| \qquad \text{for all } v.$$

In particular, if f is continuous then for all $n \in \mathbf{N}$ we have

$$\|f(e_n)\| \leqslant C\|e_n\| = C,$$

so $(\|f(e_n)\|)$ is a bounded sequence in $\mathbf{R}$.

(b) Suppose f is self-adjoint, then

$$\langle f(e_n), e_k\rangle = \langle e_n, f(e_k)\rangle \qquad \text{for all } k, n \in \mathbf{N}. \tag{$*$}$$

The left hand side is

$$\langle f(e_n), e_k\rangle = \left\langle \sum_{m=1}^{\infty} c_{nm} e_m, e_k \right\rangle = \sum_{m=1}^{\infty} c_{nm}\langle e_m, e_k\rangle = c_{nk},$$

while the right hand side is

$$\langle e_n, f(e_k)\rangle = \left\langle e_n, \sum_{m=1}^{\infty} c_{km} e_m \right\rangle = \sum_{m=1}^{\infty} \bar{c}_{km}\langle e_n, e_m\rangle = \bar{c}_{kn}.$$

We conclude that $(*)$ is equivalent to:

$$c_{nk} = \bar{c}_{kn} \qquad \text{for all } k, n \in \mathbf{N}.$$

If $(*)$ holds, it is easy to see that

$$\begin{aligned}
\langle f(v), w\rangle &= \left\langle f\left(\sum_n a_n e_n\right), \sum_k b_k e_k \right\rangle \\
&= \sum_{n,k} a_n \bar{b}_k \langle f(e_n), e_k\rangle \\
&= \sum_{n,k} a_n \bar{b}_k \langle e_n, f(e_k)\rangle \\
&= \left\langle \sum_n a_n e_n, f\left(\sum_k b_k e_k\right)\right\rangle \\
&= \langle v, f(w)\rangle.
\end{aligned}$$

Question 5. Let X be a topological space and let K, L be two subsets of X.

(a) Define the concepts: (i) X is Hausdorff; (ii) K is compact.

(b) Prove that if X is a Hausdorff topological space and K is a compact subset of X, then K is closed in X.

(c) Prove that if X is a compact topological space and $K \subseteq X$ is a closed subset, then K is compact.

(d) Suppose K and L are compact subsets of a Hausdorff topological space X. Prove that the intersection $K \cap L$ is compact.

Solution:

(a) We say that X is Hausdorff if for all $x, y \in X$ such that $x \neq y$, there exist open neighbourhoods U of x and V of y such that $U \cap V = \varnothing$.

We say that K is compact if for any open cover $\{U_i : i \in I\}$ of K, that is a collection of open subsets U_i of X such that

$$K \subseteq \bigcup_{i \in I} U_i,$$

there exists a finite subcover, that is a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$K \subseteq U_{i_1} \cup \cdots \cup U_{i_n}.$$

(b) We show that $X \setminus K$ is open. Let $x \in X \setminus K$. For each $k \in K$, since $k \neq x$ there exist open neighbourhoods U_k of k and V_k of x such that $U_k \cap V_k = \varnothing$. Putting it together we get an open cover

$$K \subseteq \bigcup_{k \in K} U_k,$$

which by compactness has a finite subcover

$$K \subseteq U_{k_1} \cup \cdots \cup U_{k_n} =: U.$$

Consider

$$V := V_{k_1} \cap \cdots \cap V_{k_n},$$

which is an open neighbourhood of x. We have $U \cap V = \varnothing$, therefore $V \subseteq X \setminus U \subseteq X \setminus K$ is an open neighbourhood of x contained in $X \setminus K$. Therefore $X \setminus K$ is open.

(c) Consider an open cover of K:

$$K \subseteq \bigcup_{i \in I} U_i.$$

We can turn this into an open cover of X:

$$X = (X \setminus K) \cup K \subseteq (X \setminus K) \cup \bigcup_{i \in I} U_i.$$

As X is compact, there is a finite subcover

$$X \subseteq (X \setminus K) \cup U_{i_1} \cup \cdots \cup U_{i_n}.$$

As $K \subseteq X$ but $K \cap (X \setminus K) = \varnothing$, we must have

$$K \subseteq U_{i_1} \cup \cdots \cup U_{i_n}.$$

(d) By part (b), K and L are closed subsets of X. Therefore, $K \cap L$ is a closed subset of K. But K is compact, so by part (c), its closed subset $K \cap L$ is also compact.

Question 6. Recall that $C_0([0,1],\mathbf{R})$ denotes the space of (bounded) continuous functions $f\colon [0,1] \longrightarrow \mathbf{R}$.

(a) State the Weierstrass Approximation Theorem for $C_0([0,1],\mathbf{R})$ with the uniform norm.

(b) Suppose $f \in C_0([0,1],\mathbf{R})$ has the property that

$$\int_0^1 f(x)\, x^n \, dx = 0 \qquad \text{for all } n = 0, 1, 2, \ldots. \tag{1}$$

Prove that f is the constant function 0 on $[0,1]$.

(c) Give an explicit **discontinuous** function $f\colon [0,1] \longrightarrow \mathbf{R}$ that satisfies equation (1) but is (obviously) not the constant function 0 on $[0,1]$.

Solution:

(a) Let $\mathcal{A}$ be the subset of $C_0([0,1],\mathbf{R})$ consisting of polynomial functions. Then $\mathcal{A}$ is dense in $C_0([0,1],\mathbf{R})$ with respect to the uniform norm.

(b) Let M be an upper bound for $|f|$ on $[0,1]$. If $M = 0$, we are done. So we may assume now that $M > 0$.

Let $\varepsilon > 0$. By the Weierstrass Approximation Theorem there exists $p \in \mathcal{A}$ such that

$$|f(x) - p(x)| < \frac{\varepsilon}{M} \qquad \text{for all } x \in [0,1].$$

Writing $p(x) = \sum_{n=0}^{d} a_n x^n$ with $a_n \in \mathbf{R}$, we have by the linearity of the integral and by the hypothesis in the question:

$$\int_0^1 f(x)p(x)\, dx = \sum_{n=0}^{d} a_n \int_0^1 f(x)\, x^n\, dx = 0.$$

Then

$$\begin{aligned} \left|\int_0^1 f(x)^2\, dx\right| &= \left|\int_0^1 f(x)\big(f(x) - p(x)\big)\, dx\right| \\ &\leqslant \int_0^1 |f(x)|\,|f(x) - p(x)|\, dx \leqslant M\, \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we conclude that the integral of the non-negative continuous function $f(x)^2$ on $[0,1]$ is zero, hence $f(x)^2$ is the constant function 0 on $[0,1]$, hence so is $f(x)$.

(c) There are many options here, but we can take for instance

$$f(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2}. \end{cases}$$

Question 7. In this question, we endow the product $S \times T$ of any two metric spaces S and T with its Manhattan metric:

$$d\big((s_1,t_1),(s_2,t_2)\big) = d_S(s_1,s_2) + d_T(t_1,t_2).$$

(a) Prove that if S and T are complete metric spaces, then the metric space $S \times T$ is complete.

(b) Define the concept of completion of a metric space X.

(c) Let X, Y be metric spaces and fix completions $\big(\widehat{X}, \iota_X\big)$ of X and $\big(\widehat{Y}, \iota_Y\big)$ of Y.

Prove that $\big(\widehat{X} \times \widehat{Y}, \iota_X \times \iota_Y\big)$ is a completion of $X \times Y$.

Solution:

(a) Let (s_n, t_n) be a Cauchy sequence in $S \times T$. I claim that (s_n) is Cauchy in S and (t_n) is Cauchy in T.

Let $\varepsilon > 0$. There exists $N \in \mathbf{N}$ such that for all $m, n \geqslant N$ we have

$$d_S(s_n, s_m) \leqslant d\big((s_n,t_n),(s_m,t_m)\big) < \varepsilon,$$

so (s_n) is a Cauchy sequence in S. Since S is complete, (s_n) converges to some $s \in S$. Similarly, (t_n) is a Cauchy sequence in T, which is complete, so (t_n) converges to some $t \in T$.

I claim that (s_n, t_n) converges to (s,t). Let $\varepsilon > 0$. There exists $N_1 \in \mathbf{N}$ such that for all $n \geqslant N_1$ we have $d_S(s_n, s) < \varepsilon/2$. There exists $N_2 \in \mathbf{N}$ such that for all $n \geqslant N_2$ we have $d_T(t_n, t) < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$, then for all $n \geqslant N$ we have

$$d\big((s_n,t_n),(s,t)\big) = d_S(s_n,s) + d_T(t_n,t) < \varepsilon.$$

(b) A completion of X is a pair $\big(\widehat{X}, \iota_X\big)$ where $\widehat{X}$ is a complete metric space and $\iota_X : X \longrightarrow \widehat{X}$ is a distance-preserving map such that $\iota_X(X)$ is dense in $\widehat{X}$.

(c) First, $\widehat{X} \times \widehat{Y}$ is complete since both $\widehat{X}$ and $\widehat{Y}$ are complete.

Let $\widehat{d}$ denote the Manhattan metric on $\widehat{X} \times \widehat{Y}$. We show that $\iota := \iota_X \times \iota_Y$ is distance-preserving:

$$\begin{aligned}
\widehat{d}\big(\iota(x_1,y_1), \iota(x_2,y_2)\big) &= \widehat{d}_X\big(\iota_X(x_1), \iota_X(x_2)\big) + \widehat{d}_Y\big(\iota_Y(y_1), \iota_Y(y_2)\big) \\
&= d_X(x_1,x_2) + d_Y(y_1,y_2) \\
&= d\big((x_1,y_1),(x_2,y_2)\big).
\end{aligned}$$

To show that the image of ι is dense in $\widehat{X} \times \widehat{Y}$, let $\big(\widehat{x}, \widehat{y}\big) \in \widehat{X} \times \widehat{Y}$ and let $\varepsilon > 0$.

Since $\iota_X(X)$ is dense in $\widehat{X}$, there exists $x \in X$ such that $\widehat{d}_X(\iota_X(x), \widehat{x}) < \varepsilon/2$. Similarly, there exists $y \in Y$ such that $\widehat{d}_Y(\iota_Y(y), \widehat{y}) < \varepsilon/2$. Then

$$\widehat{d}\big(\iota(x,y), (\widehat{x}, \widehat{y})\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Question 8. (a) Let X be a metric space.

Define the concepts: (i) X is **complete**; (ii) $f\colon X \longrightarrow X$ is a **contraction**; (iii) $x \in X$ is a **fixed point** of $f\colon X \longrightarrow X$.

Give the complete statement of the Banach Fixed Point Theorem.

(b) Consider the function $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$. Find a positive real number $a > 0$ such that f satisfies the **hypotheses** of the Banach Fixed Point Theorem on the interval $[-a, a]$.

Give your best guess for how large you can make a (without proof).

(c) What is the largest interval on which the **conclusion** of the Banach Fixed Point Theorem holds for the same function $f(x) = x^2$?

Solution:

(a) A metric space X is complete if every Cauchy sequence in X has a limit in X.

A map $f\colon X \longrightarrow X$ is a contraction if there exists $C \in [0, 1)$ such that

$$d(f(x_1), f(x_2)) \leqslant C d(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

A point $x \in X$ is a fixed point of f if $f(x) = x$.

The Banach Fixed Point Theorem: Let X be a nonempty complete metric space and let $f\colon X \longrightarrow X$ be a contraction. Then f has a unique fixed point in X. Moreover, for any choice of $x_1 \in X$, the sequence (x_n) defined by $x_{n+1} = f(x_n)$ converges to the fixed point x.

(b) Let $0 < a < \frac{1}{2}$ and let $x_1 \neq x_2 \in [-a, a]$; wlog $x_1 < x_2$. By the Mean Value Theorem applied to $f(x) = x^2$ on the interval $[x_1, x_2]$, there exists $\xi \in (x_1, x_2)$ such that

$$\begin{aligned}|f(x_2) - f(x_1)| = |f'(\xi)|\,|x_2 - x_1| = 2|\xi|\,|x_2 - x_1| &< 2a\,|x_2 - x_1| \\ &\leqslant C\,|x_2 - x_1|\end{aligned}$$

if we set $C = 2a < 1$. This tells us that f is a contraction for any $0 < a < \frac{1}{2}$. It is easy to check that f also maps $[-a, a]$ to $[-a, a]$ under the same condition, so the Banach Fixed Point Theorem can be invoked. In fact, for the function $f(x) = x^2$ we have

$$\sup_{x_1 \neq x_2 \in [-a,a]} \left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right| = \sup_{x_1 \neq x_2 \in [-a,a]} |x_1 + x_2| = 2a,$$

so the smallest constant C we can take on the interval $[-a, a]$ is $2a$, therefore the constraint $C \in [0, 1)$ forces $a < \frac{1}{2}$.

(c) I claim that the largest interval is $(-1, 1)$.

The equation $x = f(x) = x^2$ has two solutions: $x = 0$ and $x = 1$, so the uniqueness part of the conclusion gives us two possible largest intervals: $(-\infty, 1)$ or $(0, \infty)$. However, if $|x_1| \geqslant 1$ then the iteration $x_{n+1} = f(x_n)$ gives a sequence whose terms in absolute value $|x_1|^{2^n}$ go to infinity, so the sequence does not converge, which imposes the constraint $|x_1| < 1$. Intersecting this with $(-\infty, 1)$ gives $(-1, 1)$, whereas intersecting it with $(0, \infty)$ gives $(0, 1)$, which does not contain one of the solutions anymore.